MTH5114 Linear Programming and Game Theory, Spring 2024

Week 1 Coursework Questions **Viresh Patel**

These exercises should be completed individually and submitted (together with those of weeks 2 and 3) via the course QMPlus page by 9am on Monday 19 February 2024.

Make sure you clearly write your **name** and **student ID** number at the top of your submission.

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1. Say whether or not each of the following is a linear program. If it *is* a linear program, then reformulate it in standard inequality form, giving the values of the vectors $\mathbf{c}$ and $\mathbf{b}$, and the matrix A. If it is *not* a linear program, write a sentence or two explaining why.

   Note: to make your answers easier to mark, please order your vector of variables by subscript. If 2 variables have the same subscript (because you have split a variable x_i into x_i^+ and x_i^-) list x_i^+ *first* followed by x_i^-. For example: $\mathbf{x}^\mathsf{T} = (x_1, \bar{x}_2, x_3, x_4^+, x_4^-, x_5)$ is ordered as described.

   (a)
$$\begin{aligned} \text{minimize} \quad & 5x_1 + 6x_3 \\ \text{subject to} \quad & 2.9x_1 + 6x_2 + 8x_3 \geq 6.2, \\ & (x_1 - x_3)^2 \geq 16, \\ & 1.5x_1 - 18x_2 \leq 14, \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

   Solution: This is not a linear program because of the second constraint. We could replace this constraint with $|x_1 - x_3| \geq 4$, but then we are stuck because this is true if *either* $x_1 - x_3 \geq 4$ *or* $x_1 - x_3 \leq -4$, but in a linear program, a feasible solution must satisfy *all* constraints.

   (b)
$$\begin{aligned} \text{maximize} \quad & 5x_1(1 - 3x_2 + x_3) - x_2 \\ \text{subject to} \quad & x_1 + 3x_2 + x_3 \geq 4, \\ & -x_1 + x_2 - x_3 \leq 3, \\ & -2x_1 + x_2 \leq 7, \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

   Solution: This is not a linear program, because the objective function is not a linear combination of the decision variables. In particular, after expanding, we find it includes a term x_1x_2 of degree 2.

(c)

$$\begin{aligned} \text{maximize} \quad & 2x_1 + x_2 - x_3 \\ \text{subject to} \quad & 4x_1 + x_2 + 3x_3 \leq 1, \\ & -2x_2 + x_3 \leq x_1, \\ & 4x_2 + 2x_3 = -7, \\ & x_1 \text{ unrestricted}, \\ & x_2 \leq 0, \\ & x_3 \geq 0 \end{aligned}$$

Solution: This is a linear program. We replace x_1 with $x_1^+ - x_1^-$ and x_2 with $-\bar{x}_2$ (after rearranging the second constraint so all variables are on the left hand side) to obtain

$$\begin{aligned} \text{maximize} \quad & 2x_1^+ - 2x_1^- - \bar{x}_2 - x_3 \\ \text{subject to} \quad & 4x_1^+ - 4x_1^- - \bar{x}_2 + 3x_3 \leq 1, \\ & -x_1^+ + x_1^- + 2\bar{x}_2 + x_3 \leq 0, \\ & -4\bar{x}_2 + 2x_3 = -7, \\ & x_1^+, x_1^-, \bar{x}_2, x_3 \geq 0 \end{aligned}$$

Then we replace the third constraint with two inequality constraints to obtain the final equivalent standard inequality form:

$$\begin{aligned} \text{maximize} \quad & 2x_1^+ - 2x_1^- - \bar{x}_2 - x_3 \\ \text{subject to} \quad & 4x_1^+ - 4x_1^- - \bar{x}_2 + 3x_3 \leq 1, \\ & -x_1^+ + x_1^- + 2\bar{x}_2 + x_3 \leq 0, \\ & 4\bar{x}_2 - 2x_3 \leq 7, \\ & -4\bar{x}_2 + 2x_3 \leq -7, \\ & x_1^+, x_1^-, \bar{x}_2, x_3 \geq 0 \end{aligned}$$

so we have:

$$\mathbf{c} = \begin{pmatrix} 2 \\ -2 \\ -1 \\ -1 \end{pmatrix} \qquad A = \begin{pmatrix} 4 & -4 & -1 & 3 \\ -1 & 1 & 2 & 1 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & -4 & 2 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 7 \\ -7 \end{pmatrix}$$

Full credit is available for a correct final solution without intermediate steps, but it is useful to give some intermediate explanation to gain credit in case the final solution is incorrect.

Also constraints presented in a different order would give a different (but correct) matrix A and vector $\mathbf{b}$.

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Week 2 Coursework Questions **Viresh Patel**

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1. An energy company uses three different processes for generating electricity. One of the processes uses wind energy (and so requires no fuel), while the other two consume a combination of biofuel and natural gas. Each process also requires some amount of labour and emits some amount of carbon dioxide. The amount of biofuel (in Mg) and natural gas (in mcf = mega cubic feet) consumed, the labour required (in person-hours), the carbon dioxide (CO_2) emitted (in Mg), and the power generated (in MWh) per day of operation of each process is as follows:

Process	Electricity generated	CO_2 produced	Labour required	Biofuel required	Natural gas required
1	20	0	20	0	0
2	32	12	13	10	15
3	85	29	18	30	40

Each MWh of electricity can be sold at £144 and there is no limit on the amount that can be sold. Over its next planning period, the company has 320 person-hours for labour, 75 Mg of biofuel, and 90 mcf of natural gas available.

(a) The company emits all the CO_2 it produces into the atmosphere. Due to environmental regulations, they cannot emit more than 215Mg of CO_2 in this period. The company wants to know how to operate its processes to generate as much revenue as possible (you may assume that there is no limit on the number of days a process can be run in this period). Give a linear program that models this problem and state what each of your variables represents. You do not need to solve this program.

Solution:

$$\begin{aligned}
\text{maximize}\quad & 144o_2 \\
\text{subject to}\quad & i_1 = 20x_1 + 13x_2 + 18x_3, \\
& i_2 = 10x_2 + 30x_3, \\
& i_3 = 15x_2 + 40x_3, \\
& o_1 = 12x_2 + 29x_3, \\
& o_2 = 20x_1 + 32x_2 + 85x_3, \\
& i_1 \leq 320, \\
& i_2 \leq 75, \\
& i_3 \leq 90, \\
& o_1 \leq 215, \\
& x_1, x_2, x_3 \geq 0, \\
& i_1, i_2, i_3, o_1, o_2 \text{ unrestricted}
\end{aligned}$$

The variables x_1, x_2, and x_3 represent the number of days for which Plants 1, 2, and 3, respectively, operate. The variables i_1, i_2, and i_3, respectively, represent amount of labour in person-hours, Mg of biofuel, and mcf of natural gas required for this operation. The variables o_1 and o_2 represent, respectively, the total Mg of CO_2 emitted and MWh of electricity produced by this operation.

One can also substitute the unrestricted variables above to obtain a simpler solution (but no extra credit is given for this).

$$\begin{aligned}
\text{maximize}\quad & 144(20x_1 + 32x_2 + 85x_3) \\
\text{subject to}\quad & 20x_1 + 13x_2 + 18x_3 \leq 320, \\
& 10x_2 + 30x_3 \leq 75, \\
& 15x_2 + 40x_3 \leq 90, \\
& 12x_2 + 29x_3 \leq 215, \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

(b) The company decides to make use of a new carbon capture method whereby it can convert some of the CO_2 that it produces into a safe form that is not emitted. Converting 1 Mg of CO_2 in this way costs £5. The other resource constraints remain as stated. The company wants to know how to operate its processes to generate as much revenue as possible now also making use of carbon capture (you may assume that there is no limit on the number of days that the processes can operate in this period). Give a linear program that models this problem. You do not need to solve this program.

Solution: The linear programme (without substitution) is as before except we introduce a further variable c representing the amount in Mg of CO_2 that is converted into a safe form with sign restriction $c \geq 0$. We subtract $5c$ from the objective function for the extra cost of the capture. We also replace CO_2 limiting constraint $o_1 \leq 215$ with the constraint $o_1 - c \leq 215$.

Alternatively, the linear program with substitution becomes

$$\begin{aligned}
\text{maximize} \quad & 144(20x_1 + 32x_2 + 85x_3) - 5c \\
\text{subject to} \quad & 20x_1 + 13x_2 + 18x_3 \leq 320, \\
& 10x_2 + 30x_3 \leq 75, \\
& 15x_2 + 40x_3 \leq 90, \\
& 12x_2 + 29x_3 - c \leq 215, \\
& x_1, x_2, x_3, c \geq 0
\end{aligned}$$

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Week 3 Coursework Questions **Viresh Patel**

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1. For each of the following linear programs:

   (1) Sketch the feasible region of the linear program and the direction of the objective function.

   (2) Use you sketch to find an optimal solution to the program. State the optimal solution and give the objective value for this solution. If an optimal solution does not exist, state why.

   (a)

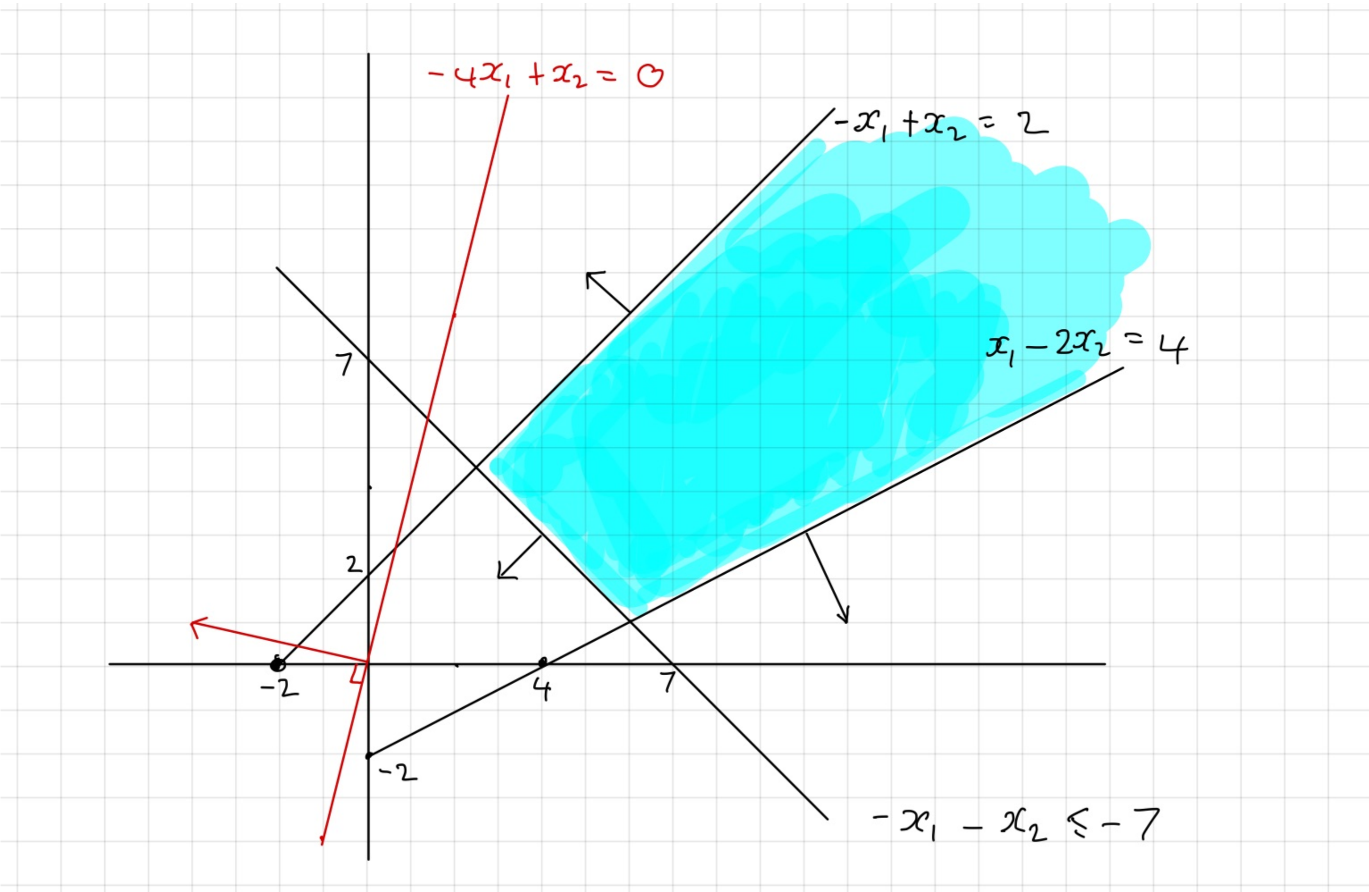
$$\begin{aligned} \text{maximize} \quad & -4x_1 + x_2 \\ \text{subject to} \quad & -x_1 + x_2 \leq 2, \\ & x_1 - 2x_2 \leq 4, \\ & x_1 + x_2 \geq 7, \\ & x_1, x_2 \geq 0 \end{aligned}$$

   (b)

$$\begin{aligned} \text{maximize} \quad & \tfrac{1}{2}x_1 + 2x_2 \\ \text{subject to} \quad & -x_1 + 2x_2 \leq 4, \\ & \tfrac{3}{2}x_1 + 3x_2 \leq 12, \\ & x_1, x_2 \geq 0 \end{aligned}$$

(a)

$$
\begin{aligned}
\text{maximize} \quad & -4x_1 + x_2 \\
\text{subject to} \quad & -x_1 + x_2 \le 2, \\
& x_1 - 2x_2 \le 4, \\
& x_1 + x_2 \ge 7, \\
& x_1, x_2 \ge 0
\end{aligned}
$$

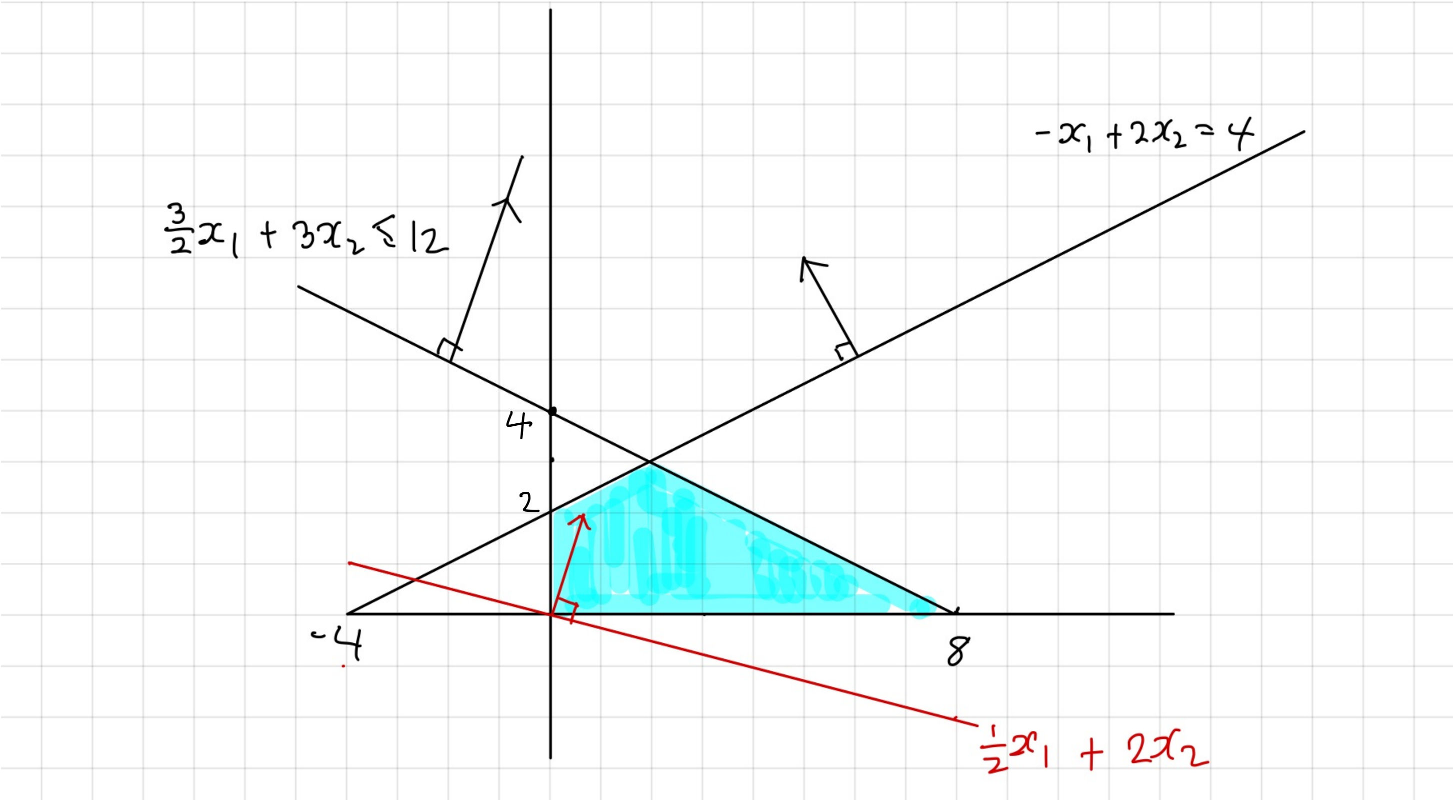


optimal solution is $(5/2, 9/2)$

The objective value is $-4 \times \frac{5}{2} + \frac{9}{2} = -\frac{11}{2}$.

(b)

$$\begin{aligned} \text{maximize} \quad & \tfrac{1}{2}x_1 + 2x_2 \\ \text{subject to} \quad & -x_1 + 2x_2 \le 4, \\ & \tfrac{3}{2}x_1 + 3x_2 \le 12, \\ & x_1, x_2 \ge 0 \end{aligned}$$



The optimal solution is (2, 3)

with objective value $\frac{1}{2} \times 2 + 2 \times 3 = 7.$