Vectors & Matrices

Problem Sheet 4

1. Let S be the sphere centred at the point $(3, -1, 6)$ with radius $6\sqrt{3}$.

   (i) Determine the Cartesian equation of the sphere S. (That is, find an equation identifying S in terms of coordinates in x, y and z).

   (ii) The set $C = \{\, (2\cos\theta + 6\sin\theta + 7\,,\ 8\cos\theta - 3\,,\ -2\cos\theta + 6\sin\theta + 2) : \theta \in [0, 2\pi) \,\}$ defines a circle in $\mathbb{R}^3$. Show that this circle lies entirely on the sphere S.

   (iii) Given that the centre of circle C is the point $(7, -3, 2)$, find the radius of C.
   (**Hint:** The radius of this circle is *not* equal to the radius of the sphere.)

   (iv) Determine the Cartesian equation of the plane that is tangent to the sphere S at the point $P = (5, 9, 8)$. (**Hint:** If the plane is tangent to S at the point P, then every vector in the plane must be orthogonal to the radial vector pointing towards P.)

2. A city's underground rail network has a *Red Line* which visits station *Portway* at coordinates $(3, -5, -5)$ and leaves in the direction of the vector $\mathbf{i} + \mathbf{j} + 6\mathbf{k}$.

   (i) Land surveyors are concerned with the proximity of this line to a new housing development at coordinates $(5, 0, 8)$. How closely does the *Red Line* approach this estate?

   (ii) The city's *Blue Line* visits the station *Queen's Road* at coordinates $(2, -8, 1)$ and leaves in the direction $2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$. The city wishes to build a tunnel connecting the *Red Line* and the *Blue Line* at their closest points. What length will this tunnel be?

   (iii) If this tunnel were completed, what distance would be traversed in order to get from *Portway* to *Queen's Road*? (You may assume that travel is only possible via the tunnels.)

3. Consider the set of quadratic functions $f(x) = ax^2 + bx + c$. For any such function, the coefficients $(a, b, c) \in \mathbb{R}^3$ can be thought of as the coordinates of a point in three-dimensional space.

   (i) Show that the subset of quadratics f with the property $f(-2) = 31$ is given by the equation $4a - 2b + c = 31$. How could this constraint be visualised in our three-dimensional space?

   (ii) A quadratic f has remainder $r(x) = x + 5$ after division by $q(x) = x^2 - 3x + 4$ if and only if it can be expressed as $f(x) = \lambda q(x) + r(x)$, for some $\lambda \in \mathbb{R}$. Express this property in terms of the coefficients a, b, c and give a geometric interpretation of this constraint.

   (iii) Find a formulation for all quadratics f with remainder $x + 5$ after division by $x^2 - 3x + 4$ such that $f(-2) = 31$.