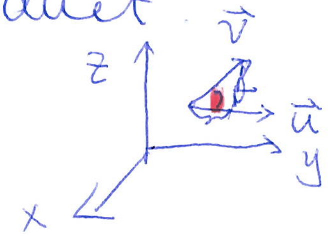
Last week: The scalar product



$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$$

5.4 Distance from a point to a Plane

recall the vector equation for a plane through point P and orthogonal to vector $\mathbf{n}$ to be

$$\mathbf{r}\cdot\mathbf{n} = \mathbf{p}\cdot\mathbf{n}$$

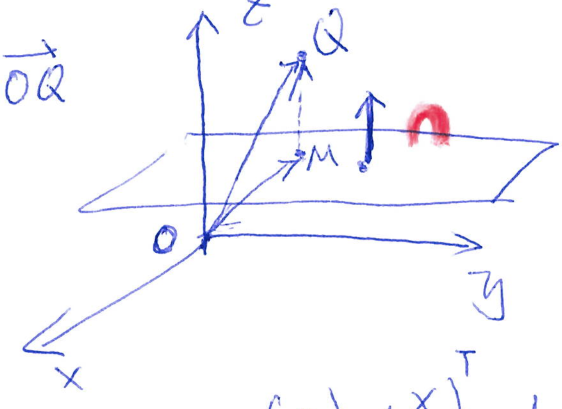
non zero vector

position vector of any point R in the plane

position vector of point P

Let Π be a plane and Q be a point with position vector $\mathbf{q}$. that is the distance from Q to the plane Π. is what we are interested.

$\mathbf{q} = \vec{OQ}$



This means the vector equation for this plane,

$$\mathbf{n}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix} = d$$

-point

$$\begin{pmatrix}a\\b\\c\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}^T = d$$

Which is equivalenty

$$ax + by + cz = d$$

The distance from Q to plane Π is the distance for Q to M where M is the point on Π which is closet Q. This means the vector reprensted by $\overrightarrow{MQ}$ is orthogonal to the plane Π. Thus. $\overrightarrow{MQ}$ is a scarlar multiple of $\mathbf{n}$

$\alpha \in R$

$$\overrightarrow{MQ} = \alpha \mathbf{n}$$

Now we can write down $|\overrightarrow{MQ}|$ by Q and $\mathbf{n}$

$$\mathbf{q} = \overrightarrow{OQ} \qquad \mathbf{m} = \overrightarrow{OM}$$

$$\mathbf{q} - \mathbf{m} = \overrightarrow{OQ} - \overrightarrow{OM} = \overrightarrow{OQ} + \overrightarrow{MO}$$
$$= \overrightarrow{MO} + \overrightarrow{OQ}$$
$$= \overrightarrow{MQ} = \alpha \mathbf{n}$$

$$\Rightarrow (\mathbf{q} - \mathbf{m}) \cdot \mathbf{n} = \alpha \mathbf{n} \cdot \mathbf{n}$$

$$\boxed{\mathbf{n} \cdot \mathbf{n} = |\mathbf{n}| \cdot |\mathbf{n}| \cdot \cos\theta}$$

$$\mathbf{q} \cdot \mathbf{n} - \mathbf{m} \cdot \mathbf{n} = \alpha |\mathbf{n}|^2$$

but M is on the plane Π so.

$$\mathbf{m} \cdot \mathbf{n} = d$$

plug this point M to the equation above

$$\mathbf{q} \cdot \mathbf{n} - d = \alpha |\mathbf{n}|^2$$

$$\alpha = \frac{\mathbf{q} \cdot \mathbf{n} - d}{|\mathbf{n}|^2}$$

Now the distance from M to Q is

$$|\overrightarrow{MQ}| = |\alpha \mathbf{n}| = \alpha \cdot |\mathbf{n}| = \frac{\mathbf{q} \cdot \mathbf{n} - d}{|\mathbf{n}|}$$

Note that if the point Q is on the plane π

$$\mathbf{q}\cdot\mathbf{n} = d$$

Thus $$|\vec{MQ}| = \frac{d-d}{|\mathbf{n}|} = 0.$$

We would also use this method to find the position vector of M, the point on plane π closest to point Q by

$$\mathbf{m} = \mathbf{q} - \alpha\mathbf{n} = \mathbf{q} - \frac{\mathbf{q}\cdot\mathbf{n} - d}{|\mathbf{n}|^2}\cdot\mathbf{n}$$

Summarising the above, we have proved the following results:

Proposition 5.4.1. If the plane π has a vector equation $\mathbf{r}\cdot\mathbf{n} = d$, and the point Q has position vector $\mathbf{q}$, then the distance between Q and π is

$$\frac{\mathbf{q}\cdot\mathbf{n} - d}{|\mathbf{n}|},$$

the point M on π closest to Q has the position vector

$$\mathbf{q} - \frac{\mathbf{q}\cdot\mathbf{n} - d}{|\mathbf{n}|^2}\cdot\mathbf{n}$$

5.5 The Vector product.

Definition 5.5.1. Given $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$.

the vector product

$\mathbf{u} \times \mathbf{v}$ is defined to be

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} u_2 \cdot v_3 - u_3 \cdot v_2 \\ u_3 \cdot v_1 - u_1 \cdot v_3 \\ u_1 \cdot v_2 - u_2 \cdot v_1 \end{pmatrix}$$

$$\begin{matrix} 2 & 3 \\ 3 & 1 \\ 1 & 2 \end{matrix}$$

In other words.

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 \cdot v_2) \cdot \mathbf{i} + (u_3 \cdot v_1 - u_1 \cdot v_3) \cdot \mathbf{j} + (u_1 \cdot v_2 - u_2 \cdot v_1) \cdot \mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = - \mathbf{v} \times \mathbf{u}$$

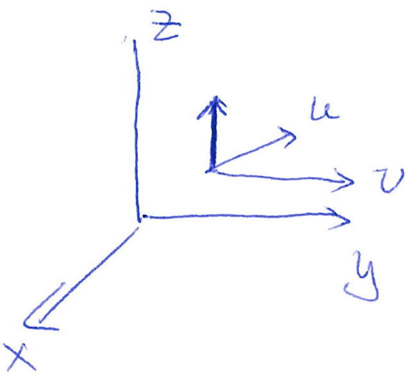
Example 5.5.2. if $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ $\mathbf{v} = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}$. then

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} 11 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \cdot 4 - (-1) \cdot 3 \\ (-1)(-1) - 1 \cdot 4 \\ 1 \cdot 3 - 2 \cdot (-1) \end{pmatrix} = \begin{pmatrix} 11 \\ -3 \\ 5 \end{pmatrix}$$

Next we want to understand the geometrical meaning of the vector product.

Proposition 5.5.3 Given vectors $\mathbf{u}$ and $\mathbf{v}$, the vector product $\mathbf{u}\times\mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, and its length $|\mathbf{u}\times\mathbf{v}|$ satisfies

$$|\mathbf{u}\times\mathbf{v}| = \begin{cases} |\mathbf{u}||\mathbf{v}|\sin\theta & \text{if } \mathbf{u}\neq\mathbf{0} \quad \mathbf{v}\neq\mathbf{0} \\ 0 & \text{if } \mathbf{u}=\mathbf{0} \text{ or } \mathbf{v}=\mathbf{0} \end{cases}$$



Proof: To prove the orthogonality we need show $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{u}=0$, and

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v}=0$$

- if $\mathbf{u}=\mathbf{0}$ or $\mathbf{v}=\mathbf{0}$, then it is straight forward that the above equations are fulfilled.
- if $\mathbf{u}\neq\mathbf{0}$ and $\mathbf{v}\neq\mathbf{0}$

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{u} = \begin{pmatrix} u_2v_3-u_3v_2 \\ u_3v_1-u_1v_3 \\ u_1v_2-u_2\cdot v_1 \end{pmatrix}\cdot\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$=(u_2v_3-u_3v_2)\cdot u_1+(u_3v_1-u_1v_3)\cdot u_2$$

$$+(u_1v_2-u_2v_1)\cdot u_3=0$$

Similarly, we can write down a equation for

$$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v}=0$$

$$|u \times v| = \sqrt{(u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2}$$

$$|u \times v|^2 = (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2$$

$$= u_2^2v_3^2 + u_3^2v_2^2 + u_3^2v_1^2 + u_1^2v_3^2 + u_1^2v_2^2 + u_2^2v_1^2$$
$$- 2(u_2v_3u_3v_2 + u_3v_1u_1v_3 + u_1v_2u_2v_1)$$

$$= u_2^2(v_3^2 + v_1^2 + v_2^2) + u_3^2(v_2^2 + v_1^2 + v_3^2)$$
$$+ u_1^2(v_3^2 + v_2^2 + v_1^2) - 2(u_2v_3u_3v_2 + u_3v_1u_1v_3 + u_1v_2u_2v_1)$$
$$- u_2^2 \cdot v_2^2 - u_3^2 \cdot v_3^2 - u_1^2 \cdot v_1^2$$

$$= (u_2^2 + u_3^2 + u_1^2)(v_3^2 + v_1^2 + v_2^2)$$
$$- (u_2^2v_2^2 + 2u_2v_3u_3v_2 + u_3^2 \cdot v_3^2)$$
$$- u_1^2 \cdot v_1^2 - 2u_1v_2u_2v_1$$
$$- 2u_3v_1u_1v_3$$

$$= |u|^2 \cdot |v|^2 - (u_2v_2 + u_3v_3)^2 - u_1^2 \cdot v_1^2$$
$$- 2u_1v_2u_2v_1$$
$$- 2u_3v_1u_1v_3$$

$$= |u|^2 \cdot |v|^2 - (u_2v_2 + u_3v_3)^2 - 2u_1v_1(u_2v_2 + u_3v_3)$$
$$- u_1^2 \cdot v_1^2$$

$$= |u|^2 \cdot |v|^2 - (u_2v_2 + u_3v_3 + u_1v_1)^2$$

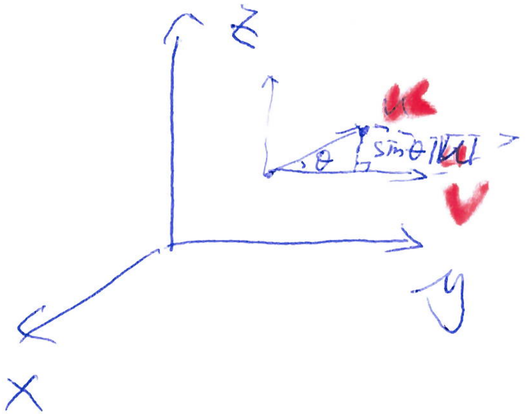
$$= |u|^2 \cdot |v|^2 - (u \cdot v)^2$$

$$= |u|^2 \cdot |v|^2(1 - \cos^2\theta) = |u|^2 |v|^2 \sin^2\theta$$

6

As a consequence of Proposition 5.5.3

we have that the $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| \cdot |\mathbf{v}| \cdot \sin\theta$,



which is the area of the parallelogram

5.6. Vector equation of a plane given 3 points on it

Let A, B, C be points in 3D-space which do not all lie on a common line. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be their position vectors. The plane containing the three points. Then

$$\mathbf{n} = (\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a}) \text{ is orthogonal}$$

to the plane Π. Thus the vector equation for the plane Π:

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}.$$

This equation

$$\underline{\underline{\mathbf{r}}} \cdot ((\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a})) = \underline{\underline{\mathbf{a}}} \cdot ((\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a}))$$

$$(\mathbf{r} - \mathbf{a}) \cdot ((\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a})) = 0$$