Queen Mary University of London

MTH5114 **Linear Programming and Games, Spring 2024**

Week 3 Seminar Questions **Viresh Patel**

Warm-Up Questions (from 2019 Midterm): Sketch each of the following linear programs in 2 variables, then find its optimal solution. If an optimal solution does not exist, say why.

$$\begin{aligned} \text{maximize} \quad & 2x_1 + x_2 \\ \text{subject to} \quad & 2x_1 + 3x_2 \leq 12, \\ & 4x_1 + 2x_2 \leq 12, \\ & x_1, x_2 \geq 0 \end{aligned}$$

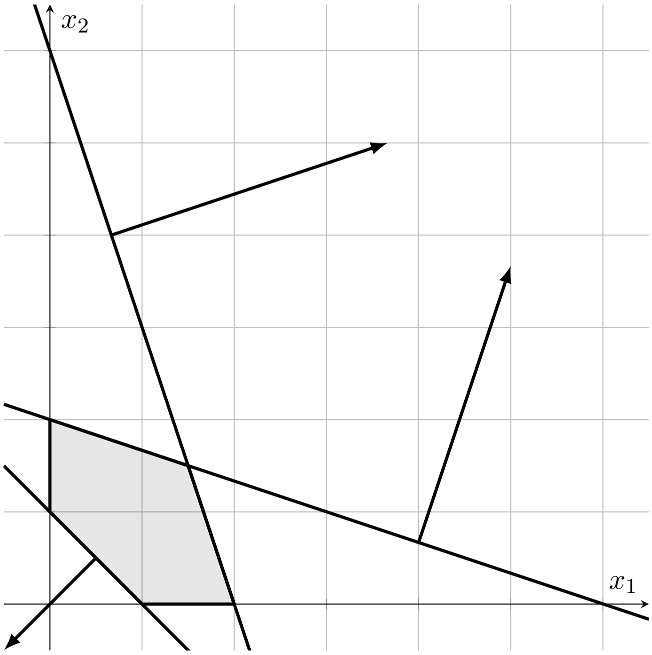
$$\begin{aligned} \text{maximize} \quad & 2x_1 - x_2 \\ \text{subject to} \quad & x_1 - x_2 \leq 1, \\ & 2x_1 - x_2 \geq 1, \\ & 2x_2 + 2x_2 \geq 4, \\ & x_1, x_2 \geq 0 \end{aligned}$$

Discussion Questions:

1. Suppose that $p, q \in \mathbb{R}$ are both constants and consider the following linear program:

$$\begin{aligned} \text{maximize} \quad & px_1 + qx_2 \\ \text{subject to} \quad & x_1 + 3x_2 \leq 6, \\ & 3x_1 + x_2 \leq 6, \\ & x_1 + x_2 \geq 1, \\ & x_1, x_2 \geq 0 \end{aligned}$$

   (a) Sketch the feasible region for these constraints.
   Solution:



(b) Find a value for the constants p and q so that $x_1 = 3/2$, $x_2 = 3/2$ is the unique optimal solution of the resulting program.

Solution: By looking at the picture, one option is to set $p = 1, q = 1$, which gives us a direction $(1, 1)$ that makes this the only optimal solution.

(c) Find a value for the constants p and q so that $x_1 = 0$, $x_2 = 1$ is the unique optimal solution of the resulting program.

Solution: Again, by looking at the picture, we see that one option is to set $p = -1, q = -0.5$.

(d) Find a value for the constants p and q for which *both* $x_1 = 3/2$, $x_2 = 3/2$ *and* $x_1 = 2$,$x_2 = 0$ are optimal solutions of the resulting program.

Solution: This can only happen if our objective points in the same direction as the normal vector of the constraint whose boundary includes these points. One such setting is to just take the normal vector itself, which gives $p = 3$, $q = 1$.

(e) Find a value for the constants p and q for which *both* $x_1 = 3/2$, $x_2 = 3/2$ *and* $x_1 = 0$,$x_2 = 2$ are optimal solutions of the resulting program.

Solution: This can only happen if our objective points in the same direction as the normal vector of the constraint whose boundary includes these points. One such setting is to just take the normal vector itself, which gives $p = 1$, $q = 3$.

(f) Determine all p and q such that $x_1 = 3/2$, $x_2 = 3/2$ is an optimal solution to the resulting program?

Solution: From the picture this happens exactly when the normal of the objective function lies "between" the normals of the first two constraints, i.e. when $\begin{bmatrix} p \\ q \end{bmatrix}$ lies "between" $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Formally, this is saying that $\begin{bmatrix} p \\ q \end{bmatrix}$ is any non-negative linear combination of the normal vectors for the first two constraints, i.e.

$$\begin{bmatrix} p \\ q \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + a_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

for any real numbers $a_1, a_2 \geq 0$. This would be an acceptable answer.
More explicitly this happens exactly when $p, q \geq 0$ with $1/3 \leq q/p \leq 3$. One way to see that this is equivalent to the above is to expand the above system of 2 equations for p and q and rearrange to get:

$$\begin{aligned} p &= a_1 + 3a_2 \\ q&3a_1 + a_2\,. \end{aligned}$$

Thus, $p, q \geq 0$ since $a_1, a_2 \geq 0$. Moreover:

$$\frac{q}{p} = \frac{3a_1 + a_2}{a_1 + 3a_2}$$

For $a_1, a_2 \geq 0$ you can check that this is an increasing function of a_1 and a decreasing function of a_2. Thus, its value will always be at least $\frac{3 \cdot 0 + a_2}{0 + 3a_2} = 1/3$ and at most $\frac{3a_1 + 0}{a_1 + 3 \cdot 0} = 3$.

2. (a) A fruit farmer has 3 hectares of apple trees and 1.5 hectares of orange trees and 4 days of work available to complete his harvest. Picking each hectare of apples takes 1 day of work and produces 0.5 truckloads of apples. Picking each hectare of oranges takes 1.5 days of work and produces 0.25 truckloads of oranges. Suppose that each truckload of apples sells for 600 pounds, and each truckload of oranges sells for 800 pounds. Write a linear program to find the most money that the farmer can make.

Solution: Let x_1, x_2 represent the number of hectares of apples and oranges, respectively, that we decide to pick. Then let a and o be the total number of truckloads of apples and oranges we obtain, and l be the total labour required. We obtain:

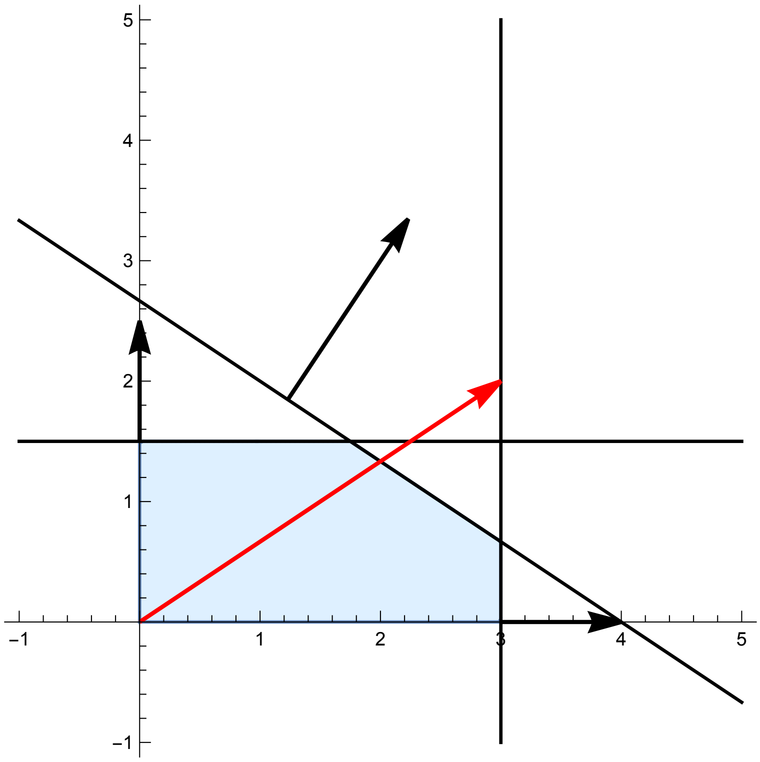
$$\begin{aligned} \text{maximize} \quad & 600a + 800o \\ \text{subject to} \quad & a = 0.5x_1, \\ & o = 0.25x_2, \\ & l = x_1 + 1.5x_2, \\ & x_1 \leq 3, \\ & x_2 \leq 1.5, \\ & l \leq 4, \\ & x_1, x_2 \geq 0, \\ & a, o, l \text{ unrestricted} \end{aligned}$$

We can simplify this by eliminating the unrestricted variables a, o, l to obtain:

$$\begin{aligned} \text{maximize} \quad & 300x_1 + 200x_2 \\ \text{subject to} \quad & x_1 + 1.5x_2 \leq 4, \\ & x_1 \leq 3, \\ & x_2 \leq 1.5, \\ & x_1, x_2 \geq 0 \end{aligned}$$

(b) Sketch you linear program and find the optimal solution. Note you may need to eliminate unnecessary variables to reduce the total number of variables to 2. Additionally, it may be helpful to change the "units" for the objective (for example, to 100's of pounds) in order to make it easier to sketch.

Solution: Using 100's of pounds for the objective (which gives us the vector $\mathbf{c}^\mathsf{T} = (3, 2)$) we get the following:



Here, the red line arrow represents the vector $\mathbf{c}$. The optimal solution is the point $\mathbf{x}^\mathsf{T} = (3, 2/3)$ with objective value $31/3$.

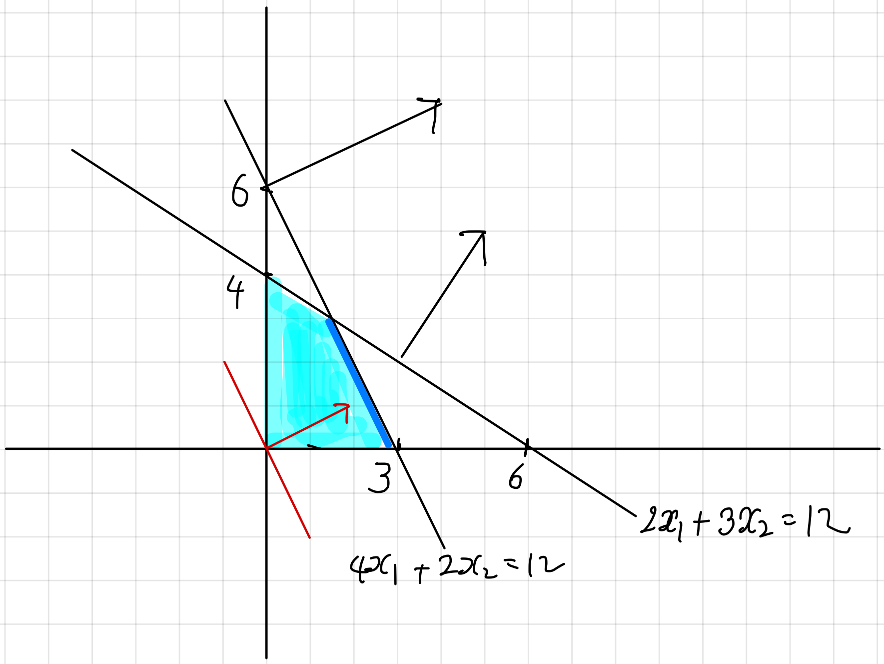
(c) For next year, should the farmer plant more apples, more oranges, or find a way to increase the number of days available (by hiring additional labour, for example)?

Solution: We can see in the picture that the optimal solution makes the constraints for hectares of apples and hours of labour tight but not the constraint for hectares of oranges (this is the horizontal line at $x_2 = 1.5$). Thus, if we want to improve the objective, we should either plant more apples or hire additional labour.

Warm-Up Questions (from 2019 Midterm): Sketch each of the following linear programs in 2 variables, then find its optimal solution. If an optimal solution does not exist, say why.

$$\begin{aligned} \text{maximize} \quad & 2x_1 + x_2 \\ \text{subject to} \quad & 2x_1 + 3x_2 \leq 12, \\ & 4x_1 + 2x_2 \leq 12, \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \text{maximize} \quad & 2x_1 - x_2 \\ \text{subject to} \quad & x_1 - x_2 \leq 1, \\ & 2x_1 - x_2 \geq 1, \\ & 2x_2 + 2x_2 \geq 4, \\ & x_1, x_2 \geq 0 \end{aligned}$$



Notice that normal to the objective function (in red) is in same direction as normal to the first constraint.

So every dark blue point is an optimal solution so infinitely many optimal solutions.

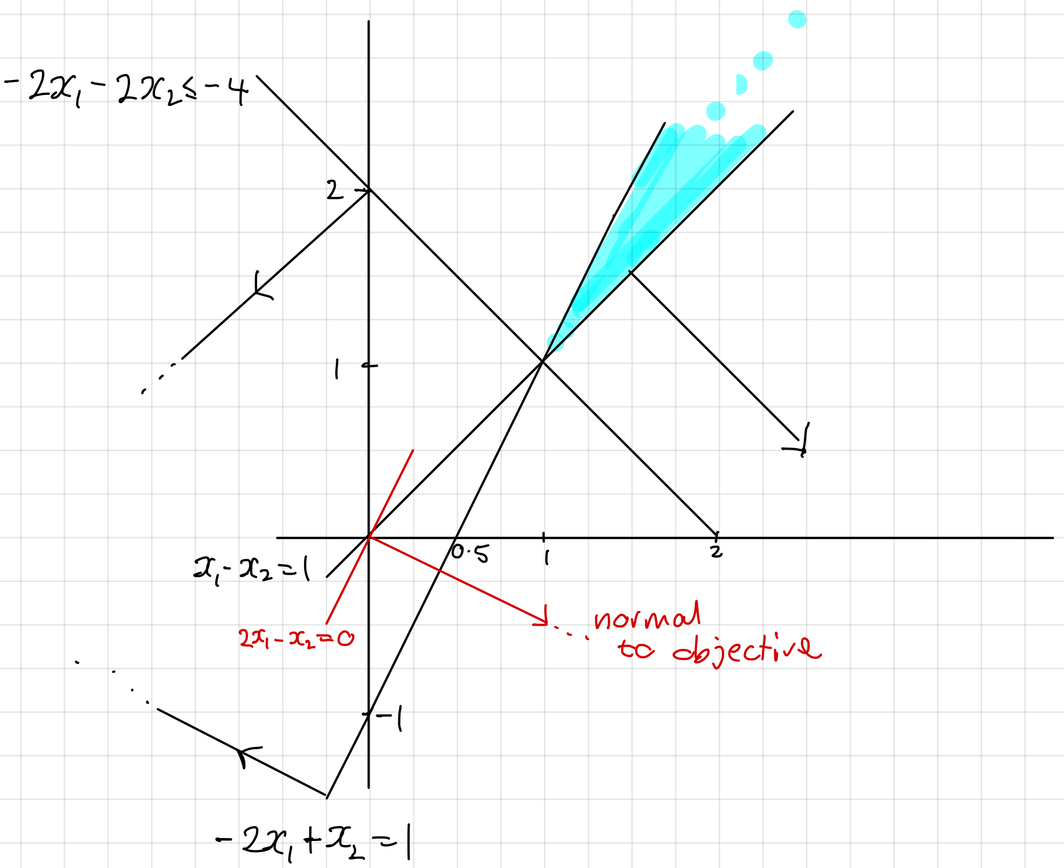
First change so all inequalities are $\leq$.

maximise $2x_1 - x_2$

subject to $x_1 - x_2 \leq 1$

$-2x_1 + x_2 \leq -1$

$-2x_2 - 2x_2 \leq -4$



This linear program is unbounded. Every line perpendicular to red arrow (sufficiently far to the right) will intersect the feasible region so the objective function can be made arbitrarily large)

(Comparing the gradients of the lines, or equivalently their normals, can help you see if the red line crosses the feasible region as it moves in the direction of the normal.