Vectors & Matrices

Solutions to Problem Sheet 2

1. (i) We proceed by deriving the Cartesian equations for the line ℓ.

We have $\mathbf{p} = \overrightarrow{OP} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, giving us the equations:

$$\frac{x-3}{1} = \frac{y-(-1)}{-2} = \frac{z-2}{3} \ ,$$

or equivalently

$$x - 3 = \frac{-y-1}{2} = \frac{z-2}{3} \ . \tag{1}$$

From the lecture notes, we can determine that any point (x, y, z) that satisfies the equations (1) lies on the line ℓ. For the point $Q = (24, -43, 65)$, we have:

$$24 - 3 = \frac{-(-43)-1}{2} = \frac{65-2}{3} = 21 \ .$$

The point Q therefore satisfies the Cartesian equations of the line ℓ, and so does indeed lie on this line.

(ii) Substituting the coordinates of $R = (1, 3, -7)$ into the first equation in (1), we get:

$$1 - 3 = -2 = \frac{-3-1}{2} \ .$$

All is fine. However, if we substitute the z-component -7 into the second equation, we get:

$$\frac{-3-1}{2} = -2 \neq -3 = \frac{-7-2}{3} \ .$$

Thus, the latter equation is not satisfied, and so the point R does not lie on the line ℓ.

(iii) In order that the point $S = (14, -23, z)$ lie on ℓ, we would need the equations

$$14 - 3 = \frac{-(-23)-1}{2} = \frac{z-2}{3}$$

be satisfied. We can see that the first equation definitely holds, so the only remaining task is to find a value $z \in \mathbb{R}$ such that:

$$\frac{z-2}{3} = 11 \ .$$

Through simple algebraic rearrangement, we find that if we take $z = 35$, the equations (1) are satisfied, and so S lies on ℓ.

2. By the formulation given in the lecture notes, we know that the set $S = \{\mathbf{p} + \lambda(\mathbf{q} - \mathbf{p}) : \lambda \in \mathbb{R}\}$ defines the line that goes through points P and Q.

   Consider the set $S_1 = \{\mathbf{p} + 2\lambda(\mathbf{q} - \mathbf{p}) : \lambda \in \mathbb{R}\}$. For any value $\lambda \in \mathbb{R}$, we can define the parameter $\mu = \frac{\lambda}{2}$. We get:

$$\mathbf{p} + \lambda(\mathbf{q} - \mathbf{p}) = \mathbf{p} + 2\mu(\mathbf{q} - \mathbf{p}) \ .$$

   Since the parameter λ in the set S_1 is a dummy variable, we can see that:

$$S = \{\mathbf{p} + \lambda(\mathbf{q} - \mathbf{p}) : \lambda \in \mathbb{R}\} = \{\mathbf{p} + 2\mu(\mathbf{q} - \mathbf{p}) : \mu \in \mathbb{R}\} = \{\mathbf{p} + 2\lambda(\mathbf{q} - \mathbf{p}) : \lambda \in \mathbb{R}\} = S_1 \ .$$

   Hence, S_1 is an equivalent formulation of the points in the line connecting P and Q.

   Similarly, take $S_2 = \{\mathbf{q} + \lambda(\mathbf{q} - \mathbf{p}) : \lambda \in \mathbb{R}\}$. For any value $\lambda \in \mathbb{R}$, we can define $\mu = \lambda - 1$ and get:

$$\begin{aligned} \mathbf{p} + \lambda(\mathbf{q} - \mathbf{p}) &= \mathbf{p} + (\mu + 1)(\mathbf{q} - \mathbf{p}) \\ &= \mathbf{p} + \mu(\mathbf{q} - \mathbf{p}) + \mathbf{q} - \mathbf{p} \\ &= \mathbf{q} + \mu(\mathbf{q} - \mathbf{p}) \end{aligned}$$

   Therefore, $S_2 = \{\mathbf{q} + \lambda(\mathbf{q} - \mathbf{p}) : \lambda \in \mathbb{R}\} = S$.

   Finally, take $S_3 = \{\mathbf{p} + \lambda(\mathbf{p} - \mathbf{q}) : \lambda \in \mathbb{R}\}$. Again, for any $\lambda \in \mathbb{R}$, we can take $\mu = -\lambda$ to get:

$$\mathbf{p} + \lambda(\mathbf{q} - \mathbf{p}) = \mathbf{p} + (-\mu)(\mathbf{q} - \mathbf{p}) = \mathbf{p} + \mu(-1(\mathbf{q} - \mathbf{p})) = \mathbf{p} + \mu(\mathbf{p} - \mathbf{q}) \ ,$$

   showing that $S_3 = \{\mathbf{p} + \lambda(\mathbf{p} - \mathbf{q}) : \lambda \in \mathbb{R}\} = S$.

   The only remaining set is $S_4 = \{2\mathbf{p} + \lambda(\mathbf{q} - \mathbf{p}) : \lambda \in \mathbb{R}\}$. This set is **not** equivalent to the other formulations of the line connecting P to Q.

To see this, we note that for the original formulation in S, if we take $\lambda = 0$, we get $\mathbf{p}$, the position vector of the point P. In this new formulation, $\lambda = 0$ gives us the vector $2\mathbf{p}$. This is the position vector of a point twice as far from the origin as P, and so, in general, the point P would no longer lie on this new line.

There is however, a condition on the vector $\mathbf{p}$ that would permit P to remain on the line. Suppose $\mathbf{p}$ and $\mathbf{q}$ faced the same direction. That is, suppose $\mathbf{p} = \lambda\mathbf{q}$ for some $\lambda \in \mathbb{R}$. In this case, the vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{q} - \mathbf{p}$ all face the same direction, and so multiplying $\mathbf{p}$ by a factor of 2 simply moves it further along the line connecting P and Q.

3. Let $V = \{\lambda\mathbf{u} + \mu\mathbf{v} : \lambda, \mu \in \mathbb{R}\}$. We define vectors $\mathbf{w_1}$ and $\mathbf{w_2}$ to be elements of the set V.

   By the definition of V, we can express $\mathbf{w_1}$ and $\mathbf{w_2}$ as:

$$\mathbf{w_1} = \lambda_1\mathbf{u} + \mu_1\mathbf{v}$$
$$\mathbf{w_2} = \lambda_2\mathbf{u} + \mu_2\mathbf{v}\ ,$$

   for some values $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$.

   We start by demonstrating the closure of V under vector addition. Consider the sum $\mathbf{w_1} + \mathbf{w_2}$. We have:

$$\begin{aligned}\mathbf{w_1} + \mathbf{w_2} &= \lambda_1\mathbf{u} + \mu_1\mathbf{v} + \lambda_2\mathbf{u} + \mu_2\mathbf{v} \\ &= (\lambda_1\mathbf{u} + \lambda_2\mathbf{u}) + (\mu_1\mathbf{v} + \mu_2\mathbf{v}) \\ &= (\lambda_1 + \lambda_2)\mathbf{u} + (\mu_1 + \mu_2)\mathbf{v} \\ &= \lambda_3\mathbf{u} + \mu_3\mathbf{v}\ ,\end{aligned}$$

   where $\lambda_3 = \lambda_1 + \lambda_2$ and $\mu_3 = \mu_1 + \mu_2$. Therefore, $\mathbf{w_1} + \mathbf{w_2} \in V$.

   We use a similar argument to prove closure under scalar multiplication. Let $\alpha \in \mathbb{R}$ be some scalar value, we have:

$$\begin{aligned}\alpha\mathbf{w_1} &= \alpha(\lambda_1\mathbf{u} + \mu_1\mathbf{v}) \\ &= \alpha\lambda_1\mathbf{u} + \alpha\mu_1\mathbf{v} \\ &= \lambda_4\mathbf{u} + \mu_4\mathbf{v}\ ,\end{aligned}$$

   where $\lambda_4 = \alpha\lambda_1$. Therefore, $\alpha\mathbf{w_1} \in V$.

Note that none of these arguments actually required us to use the coordinates of the vectors $\mathbf{w_1}$ and $\mathbf{w_2}$. The statement in the question holds for **any** pair of vectors in $\mathbb{R}^3$.

4. (i) By Theorem 5.1.3 in the lecture notes, we have:

$$\mathbf{u} \cdot \mathbf{v} = 4 \cdot 1 + (-1) \cdot (-9) + 4 \cdot 2 = 4 + 9 + 8 = 21 \ .$$

(ii) Since we are able to express $\mathbf{v}$ and $\mathbf{w}$ in terms of their coordinates, we can take their sum by simply summing their individual components:

$$\mathbf{v} + \mathbf{w} = \begin{pmatrix} 1 \\ -9 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -8 \\ 2 + z \end{pmatrix} .$$

Again, by Theorem 5.1.3, we have:

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = 4 \cdot 3 + (-1) \cdot (-8) + 4 \cdot (2 + z) = 12 + 8 + 8 + 4z = 28 + 4z \ .$$

In order for $\mathbf{u}$ to be orthogonal to $\mathbf{v} + \mathbf{w}$, we must have $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = 0$. Therefore, our condition on z becomes $28 + 4z = 0$, giving us $z = -7$.