RELATIVITY – MTH6132

PROBLEM SET 4

1. The 4–velocity $\overline{U} = (U^0, U^1, U^2, U^3)$ corresponds to the 3–velocity, $\underline{v}$, in the sense that $\overline{U} = \gamma(v)(1, \underline{v})$. Express

(a) U^0 in terms of $|\underline{v}|$

(b) U^α in terms of $\underline{v}$, where α represents the spatial components and takes values ($\alpha = 1, 2, 3$)

(c) U^0 in terms of U^α

(d) $\frac{d}{d\tau}$ in terms of $\frac{d}{dt}$ and $\underline{v}$, where τ is proper time

(e) v^α in terms of U^α

(f) $|\underline{v}|$ in terms of U^0

2. A particle at rest with mass m_0 is hit by a photon with frequency ν. The photon is absorbed, and the resulting particle has a new mass m and moves away with speed u. Thus, $\bar{P}_\gamma + \bar{P}_0 = \bar{P}$, where $\bar{P}_\gamma$, $\bar{P}_0$ and $\bar{P}$ are the 4-momenta of the photon, initial particle and resulting particle respectively.

(a) Using $|\bar{P}|^2 = |\bar{P}_\gamma + \bar{P}_0|^2$, find m in terms of m_0 and ν.

(b) Using $|\bar{P}_0|^2 = |\bar{P}_\gamma - \bar{P}|^2$, and the result of part (a), show that

$$\frac{1-u}{1+u} = \frac{m_0}{m_0 + 2h\nu}.$$

Hence find u in terms of m_0 and ν.

3. An atom of rest mass m_0 at rest in a laboratory absorbs a photon of frequency ν. Use the conservation law of 4-momentum to find the velocity and rest mass of the resulting particle.

4. A stationary particle of rest mass M decays into a massive particle with rest-mass m_1 moving with speed u_1 towards the left along the x-axis and a massless particle moving in the opposite direction with energy E. Show that

$$E = \frac{m_1 u_1}{\sqrt{1 - u_1^2}}.$$

Use this to derive an expression relating the rest-mass M directly to m_1 and u_1.

5. A positron with velocity 4/5 is anhilated in a collision with a stationary electron, yielding two photons which emerge in opposite directions along the track of the incoming particle. If m_0 is the mass of the electron and positron, find the energies of the photons.

6. A particle has rest mass m_0. Whilst at rest, it emits a photon and, as a result, its rest mass is reduced to $m_0/2$.

(a) By comparing components of the 4-momenta before and after the event, show that the speed of the particle after the reduction of mass is $\frac{3}{5}$.

(b) Show that the energy $E = h\nu$ of the photon is $3m_0/8$.

7. A particle of rest mass M moving along the x-axis with speed v decays into two particles each with a rest mass $M/2$. Both particles continue to move along the x-axis. Show that the new particles move with the same speed. Show also that the speed of the new particles equals that of the original particle.

8. A stationary particle of rest mass M decays into two particles that both move along the x-axis. One has a rest-mass m_1 and a speed u_1 and the other has a rest-mass m_2 and a speed u_2. Prove that

$$M^2 = m_1^2 + m_2^2 + 2m_2^2 \frac{u_2}{1 - u_2^2} \left(\frac{1}{u_1} + u_2 \right).$$

9. A particle moving along the x–axis with speed v disintegrates into two photons (particles with zero rest mass) moving in directions making angles α and β with the x–axis and on opposite sides of it. Show that

$$v = \frac{\sin\beta\cos\alpha + \sin\alpha\cos\beta}{\sin\alpha, + \sin\beta}$$

where we assume the speed of light $c = 1$. Show also that

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{1 - v}{1 + v}.$$

Conclude that if a photon disintegrates into two photons, they must both move along the direction of the original photon.

Hint: For the second part of this question use the identities:

$$\sin 2\alpha = 2\sin\alpha\cos\alpha, \quad \sin(\alpha/2) = \sqrt{\frac{1}{2}(1 - \cos\alpha)}, \quad \cos(\alpha/2) = \sqrt{\frac{1}{2}(1 + \cos\alpha)}.$$

10. Consider the collision of a proton and an anti-proton both with rest mass m_p and 3-velocity v in units of $c = 1$, see Fig. 1.

(a) Assuming that the collision is head on along the x axis so that the 3-velocities of the proton and the anti-proton are v and $-v$ respectively, find the value of v so that the collision produces a Higgs boson at rest with rest mass $M_H = 125\, m_p$.

(b) Consider now that both the proton and the anti-proton move along the positive x-direction forming an angle θ_1 and θ_2 respectively with respect to the x-axis in the $x - y$ plane. See the Figure below. Find v and θ_1 such that the Higgs boson that is formed as a result of the collision moves only along the positive x-direction with 3-velocity $u = \frac{\sqrt{3}}{2}$.

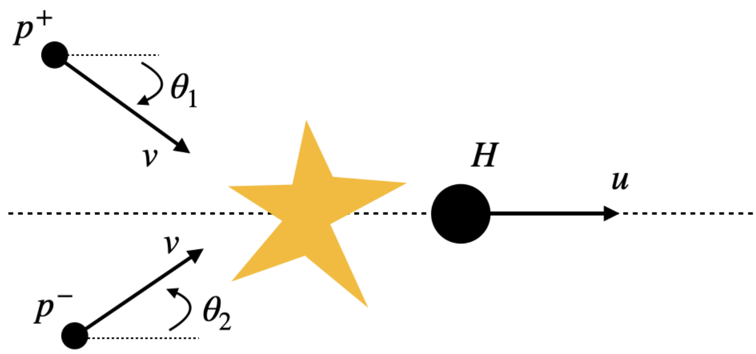

Figure 1: Collision of a proton and an anti-proton.

11. Consider two vector fields X and Y and an arbitrary smooth scalar function $f(x)$. The Lie derivative of f along X is given by $\mathcal{L}_X(f) = X(f) = X^a\partial_a f$. Similarly, in the lectures we saw that the Lie derivative of Y along X, written as $\mathcal{L}_X Y$, is given by the commutator $[X, Y]$. From the definition of $[X, Y]$ acting on f as in the notes,

$$[X, Y](f) \equiv X(Y(f)) - Y(X(f))\,,$$

1. Show that the components of the vector field $[X, Y]^a$ are

$$[X, Y]^a = X^b\partial_b Y^a - Y^b\partial_b X^a\,.$$

2. Show that $[X, Y]^a$ transforms as a vector under general coordinate transformations.

3. Consider a one form ω_a. Using that the Lie derivative satisfies the Leibniz rule and recalling that $\omega_a Y^a$ is a scalar, show that

$$\mathcal{L}_X(\omega)_a = X^b\partial_b\omega_a + \omega_b\partial_a X^b\,.$$

4. Is $\mathcal{L}_X(\omega)_a$ a one form?

Reading Assignment & Further Exploration: Relativistic Doppler Effect.

- Read carefully the subsection on Doppler shift in the Week 4 Lecture Notes, checking all the calculations (and asking me if you have any questions).
- Compare and contrast these results with the classical Doppler effect, discussed in the Further Exploration exercises of Problem Set 3.
- For further exploration, use the Suggested Reading list (found in the module syllabus) to learn how relativistic Doppler shift is used in Cosmology and write one-page summary of your findings.