Planes and vector product

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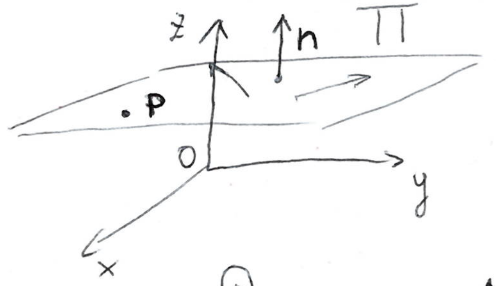
February 2024

Distance from a point to a plane



$r = (x, y, z)$

$p = (p_1, p_2, p_3)$

$n = (a, b, c)$

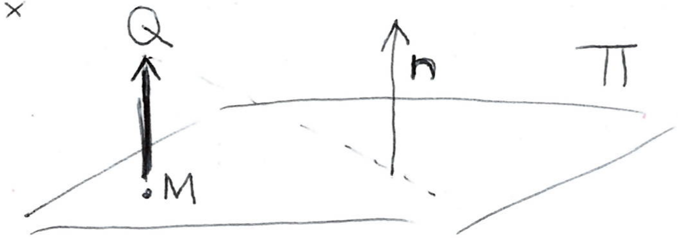
$(r - p) \cdot n = 0$

$(x - p_1, y - p_2, z - p_3) \cdot n = 0$

$a(x - p_1) + b(y - p_2) + c(z - p_3) = 0$

$ax + by + cz = \underbrace{ap_1 + bp_2 + cp_3}$

$\boxed{ax + by + cz = d}$ $\quad a, b, c, d \in \mathbb{R}$



$Q \notin \Pi$

$|\overrightarrow{MQ}| > 0$

$M \in \Pi$

$\overrightarrow{OM} = \mathbf{m}$

$\boxed{r \cdot \mathbf{n} = d}$ $\quad \Pi$

$\boxed{m \cdot n = d}$ $\quad M \in \Pi$

$\boxed{q - m = \alpha n}$, $\quad \alpha \in \mathbb{R}$

$|n| = \sqrt{a^2 + b^2 + c^2}$

$|\overrightarrow{MQ}| = |q - m| = |\alpha \mathbf{n}| = |\alpha|\,|\mathbf{n}|$

$(q - m) \cdot n = \alpha n \cdot n$

$q \cdot n - m \cdot n = \alpha |n|^2$

$q \cdot n - d = \alpha |n|^2$

$|n| \neq 0$

$$\boxed{\alpha = \frac{q \cdot n - d}{|n|^2}}$$

$$|\overrightarrow{MQ}| = \frac{|q \cdot n - d|}{|n|^2}\,|n| = \frac{|q \cdot n - d|}{|n|}$$

Proposition 5.4.1

If the plane Π has equation $\mathbf{r} \cdot \mathbf{n} = d$, and the point Q has position vector $\mathbf{q}$, then the distance between Q and Π is

$$\frac{|\mathbf{q} \cdot \mathbf{n} - d|}{|\mathbf{n}|},$$

and the point on Π that is closest to Q has position vector

$$\mathbf{q} - \left(\frac{\mathbf{q} \cdot \mathbf{n} - d}{|\mathbf{n}|^2} \right) \mathbf{n}\,.$$

$$\overrightarrow{MQ} = q - m = q - \alpha n = q - \frac{q \cdot n - d}{|n|^2} n$$

Π $\quad x - y + 2z = 3 \quad n = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \quad d = 3 \quad q \cdot n = 1 + 0 - 2 = -1$

$|n| = \sqrt{1+1+4} = \sqrt{6}$

$Q = (1, 0, -1) \quad q = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \mathrm{dist}(Q, \Pi) = \frac{|q \cdot n - d|}{|n|} = \frac{|-1-3|}{\sqrt{6}} = \frac{4}{\sqrt{6}}$

The vector product

Definition 5.5.1

Given vectors $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, the *vector product* $\mathbf{u} \times \mathbf{v}$ is defined to be

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}.$$

$u \; v$

$u \times v \overset{?}{=} v \times u$ No

In other words, $i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $j = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $k = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\mathbf{i} - (u_1 v_3 - u_3 v_1)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}.$$

$u = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ $v = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}$

$u \times v = \begin{pmatrix} 11 \\ -3 \\ 5 \end{pmatrix}$

$v \times u = \begin{pmatrix} -11 \\ 3 \\ -5 \end{pmatrix}$

$$u \times v = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$$

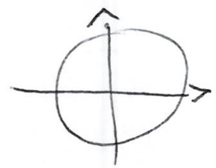
$$|u \times v|^2 = (u_2 v_3 - u_3 v_2)^2 + (u_3 v_1 - u_1 v_3)^2 + (u_1 v_2 - u_2 v_1)^2$$

$$= u_2^2 v_3^2 + u_3^2 v_2^2 - 2 u_2 v_3 u_3 v_2$$
$$+ u_3^2 v_1^2 + u_1^2 v_3^2 - 2 u_3 v_1 u_1 v_3$$
$$+ u_1^2 v_2^2 + u_2^2 v_1^2 - 2 u_1 v_2 u_2 v_1$$

Check the formula as an extra problem

$$= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (\underbrace{u_1 v_1 + u_2 v_2 + u_3 v_3}_{})^2$$

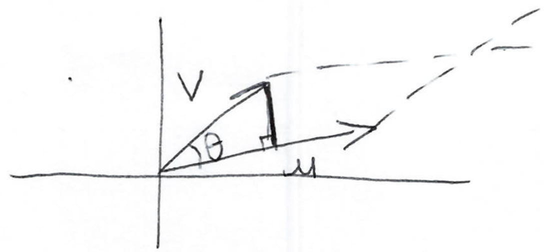
$$|u \times v|^2 = |u|^2 |v|^2 - (u \cdot v)^2$$
$$= |u|^2 |v|^2 - (|u|\,|v| \cos\theta)^2$$



$$|u \times v|^2 = |u|^2 |v|^2 (1 - \cos^2\theta) = |u|^2 |v|^2 \sin^2\theta$$

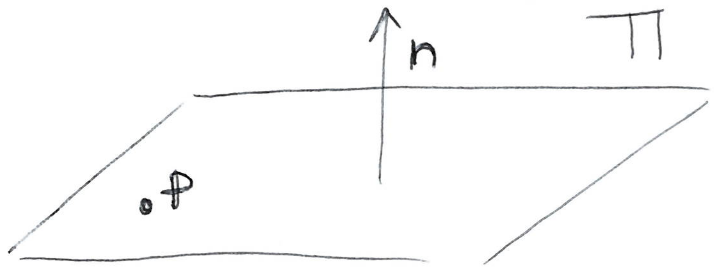
$$0 \leq \theta \leq \pi$$

$$\Longrightarrow \quad |u \times v| = |u|\,|v| \sin\theta$$



Area of the parallelogram
$|u \times v| = |u|\,|v| \sin\theta$

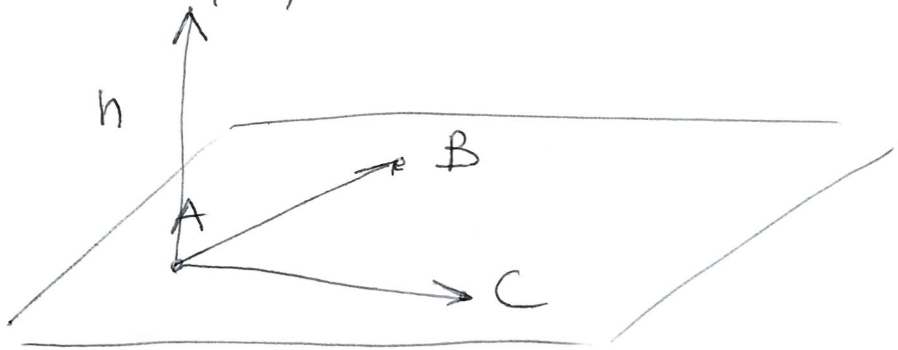
Vector equation of a plane given 3 points on it



$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\quad (r - p) \cdot n = 0$

$n = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ $\quad ax + by + cz = d$

A, B, C points on Π



$u = b - a$

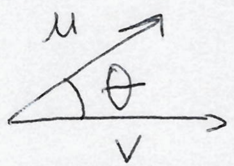
$v = c - a$

$n = u \times v = \underline{(b - a) \times (c - a)}$

Π $\quad (r - a) \cdot \big((b - a) \times (c - a) \big) = 0$

Proposition 5.5.3

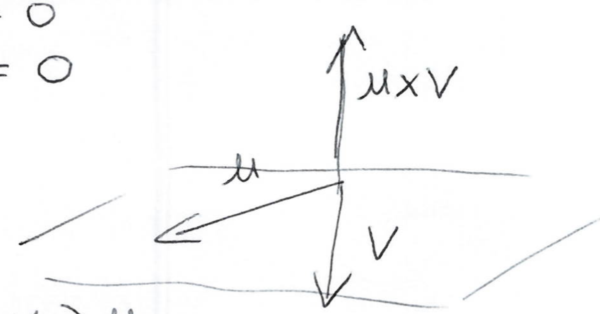
Given vectors **u** and **v**, the vector product $\mathbf{u} \times \mathbf{v}$ is orthogonal to both **u** and **v**, and its length $|\mathbf{u} \times \mathbf{v}|$ satisfies



$$|\mathbf{u} \times \mathbf{v}| = \begin{cases} |\mathbf{u}||\mathbf{v}| \sin\theta & \text{if } \mathbf{u} \neq \mathbf{0}, \mathbf{v} \neq \mathbf{0} \\ 0 & \text{if } \mathbf{u} = \mathbf{0} \text{ or } \mathbf{v} = \mathbf{0} \end{cases}$$

where θ denotes the angle between **u** and **v** (in the case that they are both non-zero).

Proof I need to check that $(u \times v) \cdot u = 0$
$(u \times v) \cdot v = 0$



$$(u \times v) \cdot u = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} =$$

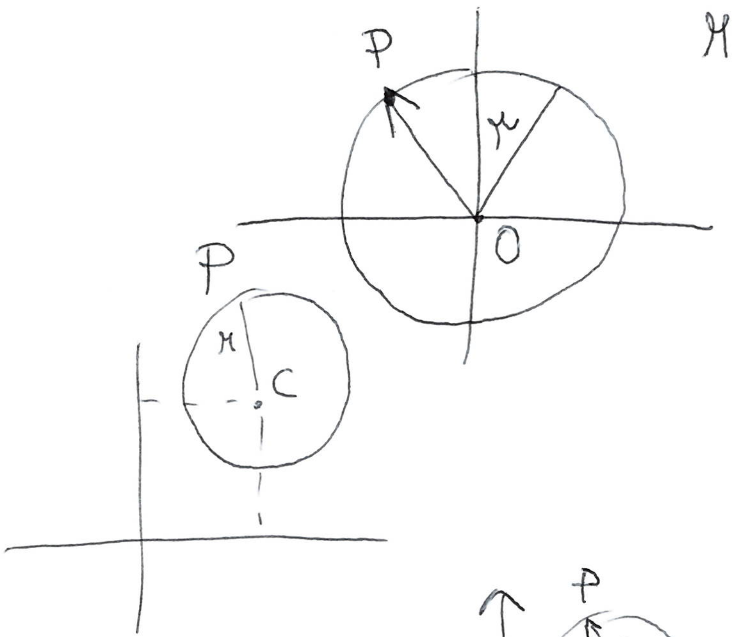
$$= (u_2 v_3 - u_3 v_2) u_1 + (u_3 v_1 - u_1 v_3) u_2 + (u_1 v_2 - u_2 v_1) u_3$$

$$= \cancel{u_2 v_3 u_1} - \cancel{u_3 v_2 u_1} + \cancel{u_3 v_1 u_2} - \cancel{u_1 v_3 u_2} + \cancel{u_1 v_2 u_3} - \cancel{u_2 v_1 u_3} = 0$$

$(u \times v) \cdot v = 0$

$u \neq 0$
$v \neq 0$

Equation of a circle in $\mathbb{R}^2$ and the sphere in $\mathbb{R}^3$

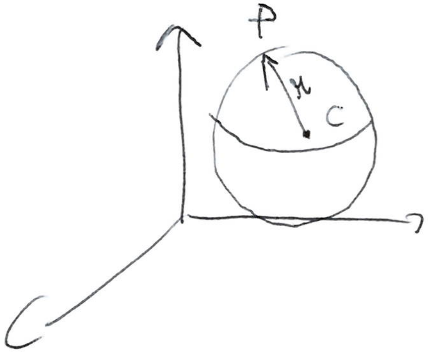


$r > 0 \qquad P = (x, y)$

$$|\overrightarrow{OP}| = r \iff |\overrightarrow{OP}|^2 = r^2$$

$$\iff x^2 + y^2 = r^2$$

$$|\overrightarrow{CP}|^2 = r^2 \iff (x - x_C)^2 + (y - y_C)^2 = r^2$$



$r > 0$

$$|\overrightarrow{CP}|^2 = r^2 \iff$$

$$(x - x_C)^2 + (y - y_C)^2 + (z - z_C)^2 = r^2$$

$$S^n = \{x = (x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : |x| = 1\}$$

$$\sum_{i=1}^{n+1} x_i^2 = 1$$

Distance from a point to a line

Distance between two lines

Intersection of planes and systems of linear equations

Intersection of other geometric objects