Vectors & Matrices

Problem Sheet 3

1. Using the Cauchy-Schwarz Inequality, prove that for any vector $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, we have

$$|u_1| + |u_2| + |u_3| \leq \sqrt{3}|\mathbf{u}| \ .$$

2. (i) Let L be the set of points on the line-segment between vertices $(1, 0)$ and $(0, 1)$. By considering the set of vectors $\{\, \lambda\mathbf{i} + (1 - \lambda)\mathbf{j} \, : \, 0 \leq \lambda \leq 1 \,\}$, show that the maximal distance between the origin and the line-segment L is 1.

   (ii) Use your result from part (i) to show that for any square with sides of length 1, the maximal distance between any two points on the square is $\sqrt{2}$.

   (iii) Show that for any two squares with side-length 1 and a non-empty intersection, the maximal distance between two points within their union is $2\sqrt{2}$.

3. The plane Π contains the point $P = (2,\, 2, -2)$, and is such that the vector $\mathbf{n} = 6\mathbf{i} + \mathbf{j} + 5\mathbf{k}$ is orthogonal to its surface.

   (i) Determine the Cartesian equation that defines the plane Π.

   (ii) Consider the set of points $C = \{\, (\lambda,\ \lambda^2,\ 4) \, : \, \lambda \in \mathbb{R} \,\}$. C defines a parabola in three-dimensional space. Find the minimal distance between the parabola C and the plane Π.

4. Find the vectors that result from the following vector products:

$$\text{(i)} \quad \mathbf{i} \times \mathbf{j} \ , \quad \text{(ii)} \quad \mathbf{j} \times \mathbf{i} \ , \quad \text{(iii)} \quad \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} \ , \quad \text{(iv)} \quad \begin{pmatrix} 17 \\ 119 \\ -53 \end{pmatrix} \times \begin{pmatrix} 17 \\ 119 \\ -53 \end{pmatrix} \ .$$

5. The set of points $C = \{\, (\ 15\sin\theta + 4\,,\ \cos\theta - 4\sin\theta\,,\ 2\cos\theta + 2\sin\theta\ ) \ : \ \theta \in [0, 2\pi) \,\}$ defines a circle in $\mathbb{R}^3$.

   (i) Identify three points on the circle C, and determine the Cartesian equation that defines the plane containing them.

   (ii) Find the position vector of the point on this plane that lies closest to the origin.

   (iii) Prove that *all* points on C fall within this plane.

   (iv) Show that the point $(1, -1,\, 1)$ does not lie on the circle C.