MTH5114: Linear Programming and Games **Spring 2024**

Week 2 Seminar Questions **Viresh Patel**

Past Exam Questions:

1. **From 2019 Exam:** A clothing company produces two different types of shirts: casual shirts, and dress shirts. Each shirt requires some amount of fabric (measured in metres), buttons, and labour (measured in hours). The company has production lines for producing each type of shirt separately. Additionally, the company has determined that it can waste less fabric if it cuts parts for both types of shirts out of the same length of cloth. It has developed a third production line for this method, which costs slightly more labour.

   The total amount of resources required and the number of shirts produced by using each production line for one day is shown below:

Production Line	Fabric Required	Buttons Required	Labour Required	Casual Shirts Produced	Dress Shirts Produced
1	140	210	85	70	0
2	125	400	75	0	50
3	200	500	180	60	40

   Suppose that the company has only 1500 metres of fabric, 3500 buttons, and 1800 hours of labour available. It can sell an unlimited number of casual shirts for £45 each and an unlimited number of dress shirts for £65 each. It wants to know the most revenue it can generate using only these available resources. Write a linear program that models this problem (you do not need to solve this program).

   Solution: Let x_i (for $i = 1, 2, 3$) be the number of days that production line i is run; these are our decision variables. The variables f, b, l, c, d represent respectively the length in meters of fabric required, the number of buttons used, the number of hours of labour required, the number of casual shirts produced, and the number of dress shirts produced.

$$
\begin{aligned}
\text{maximize} \quad & 45c + 65d \\
\text{subject to} \quad & f = 140x_1 + 125x_2 + 200x_3, \\
& b = 210x_1 + 400x_2 + 500x_3, \\
& l = 85x_1 + 75x_2 + 180x_3, \\
& c = 70x_1 + 60x_3, \\
& d = 50x_2 + 40x_3, \\
& f \leq 1500, \\
& b \leq 3500, \\
& l \leq 1800, \\
& x_1, x_2, x_3 \geq 0, \\
& f, b, l, c, d \text{ unrestricted}
\end{aligned}
$$

2. A major supermarket chain has decided to sub-contract production of ready-to-eat sandwiches to your production plant. Your plant can make three kinds of sandwiches, each of which take the following amounts of raw ingredients (measured in grams) to produce:

Sandwich	Chicken	Cheese	Lettuce	Tomato	Bread
Chicken Salad	45	0	15	15	60
Chicken Club	25	20	10	10	90
Cheese and salad	0	30	20	13	60

The total amount of time (measured in minutes) to make one of each sandwich is listed below, together with the price that the supermarket will pay to purchase one of each:

Sandwich	Time to Make	Price
Chicken Salad	2.8	£1.25
Chicken Club	4.8	£1.65
Cheese and Salad	1.3	£0.95

For the sake of this problem you can treat ingredients used and number of sandwiches made as continuous quantities—we could perhaps think of this as the *average* made per day over a longer period, in which case making 100.43 sandwiches per day would still make sense.

(a) Suppose that you have 5000g of chicken, 2500g of cheese, 1500g of lettuce, 6000g of bread, and 500g of tomato available from your supplier each day, and a total of 120 person-hours (= 7200 total minutes) of labour available each day. Write a linear program to find a production plan that maximises your daily revenue.

Solution: Let our decision variables x_1, x_2, x_3 be the number of chicken salad, chicken club, and cheese and salad sandwiches made respectively. Also let k, c, l, m, b, t represent the total amounts of chicKen, Cheese, Lettuce, toMato, Bread, and Time used for production. Then, we have:

$$\begin{aligned}
\text{maximize} \quad & 1.25x_1 + 1.65x_2 + 0.95x_3 \\
\text{subject to} \quad & k = 45x_1 + 25x_2, \\
& c = 20x_2 + 30x_3, \\
& l = 15x_1 + 10x_2 + 20x_3, \\
& m = 15x_1 + 10x_2 + 13x_3, \\
& b = 60x_1 + 90x_2 + 60x_3, \\
& t = 2.8x_1 + 4.8x_2 + 1.3x_3, \\
& k \leq 5000, \\
& c \leq 2500, \\
& l \leq 1500, \\
& m \leq 500, \\
& b \leq 6000, \\
& t \leq 7200, \\
& x_1, x_2, x_3 \geq 0, \\
& k, c, l, m, b, t \text{ unrestricted}
\end{aligned} \tag{1}$$

You could also give a program without the additional variables k, c, l, m, b, t. Its value is what you get if you substitute the equations for each of these variables into the objective function and the other constraints.

(b) Suppose now that everything is exactly as in the previous question, but now (due an agreement with a supplier) you can *purchase* additional lettuce beyond the 1500g you have available at a cost of £0.03 per gram and, conversely, if you use less than 1500g you can *sell* any you have left over for the same price of £0.03 per gram.Describe how to modify your program from the previous question to calculate the best profit.

Note: when calculating profit, you may ignore the cost of the other raw ingredients already on hand (suppose we have already purchased them, and so their costs have already been fixed). However, you *should* include the cost or profit from buying extra lettuce (beyond the 1500g you already have) or selling lettuce (leftover from the 1500g you already have).

Solution: We can introduce a new variable L, which is the amount of lettuce used beyond the 1500g available so that $L = l - 1500$ and a negative values of L indicates the amount of lettuce left unused that can be sold. Now, we can remove the constraint $l \leq 1500$ and subtract $0.03L$ from the objective function and declare that L is unrestricted.

Alternatively, we can simply remove the constraint on l from the program, and add an extra term $0.03 \cdot (1500 - l)$ to our objective. Note this captures both revenue (if we use $l < 1500g$ of lettuce) and costs (if we use $l > 1500g$ of lettuce).

Questions for Discussion and Review:

1. This question is to test basic understanding of feasible solutions and optimal solutions. Give an example of a linear programme that satisfies all of the following properties:

   - the linear programme has 3 variables and 2 constraints
   - the linear program is in standard inequality form
   - $\begin{pmatrix}1\\3\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\\4\end{pmatrix}$, are feasible solutions
   - $\begin{pmatrix}1\\3\\0\end{pmatrix}$ is **not** an optimal solution (and explain why this is the case for your example).

   Note that there are infinitely many correct answers to this question. What is the "simplest" example you can come up with?

   Solution: The following is such an example

$$\begin{aligned} \text{maximize} \quad & x_1 + x_2 + x_3 \\ \text{subject to} \quad & x_1 + x_2 \leq 5, \\ & x_2 + x_3 \leq 6, \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

   $(1, 3, 0)^T$ is not optimal because its objective value is $1+3+0 = 4$, whereas $(0, 1, 4)^T$ has objective value $0 + 1 + 4 = 5$ which is larger. (Both are feasible solutions since they satisfy all constraints and sign restrictions).

   A fairly trivial example that satisfies the properties is

$$\begin{aligned} \text{maximize} \quad & x_3 \\ \text{subject to} \quad & x_3 \leq 4, \\ & x_3 \leq 5, \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

2. Have a look again at Q3 and 4 from the previous week's seminar questions now that you have seen the definitions of feasible solution and optimal solution.

3. In lectures, we saw an example of a multi-stage production problem. What about a multi-stage transportation problem? Suppose a petroleum company wants to ship crude oil from three locations, using some intermediate ports, at which shipments can be combined or divided before being sent onward. The overall shipping network is depicted below. We have labelled each link with the cost to ship 1 L across it, as well as the maximum amount (in L) that can be shipped across it:

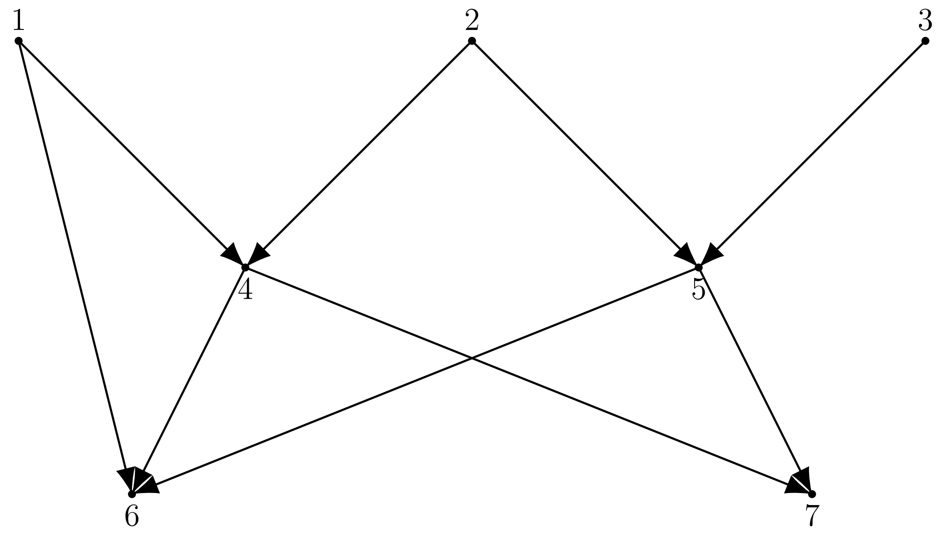

Table 1: Cost per kL

	4	5	6	7
1	20		50	
2	35	40		
3		42		
4			13	62
5			58	19

Table 2: Capacity in kL

	4	5	6	7
1	2100		3000	
2	1800	2400		
3		5100		
4			1700	4200
5			1900	5300

Suppose that there are 4000, 2100, and 3000 kL of crude oil available at locations 1,2, and 3, respectively, and that the company must ship a total of 3100 kL to location 6 and 6000 kL to location 7. Write a linear program to find a shipping plan that accomplishes this as cheaply as possible while respecting the capacity constraints for each route.

Hint: It might be useful to start by writing constraints to first ship all the available oil out along the routes leaving 1,2, and 3, and then think about what constraints should be added to find a feasible shipment of oil out of 4 and 5.

Solution: We can model this by introducing a variable along each route, just as in the standard case. Let $x_{i,j} \geq 0$ be how many units we ship from location i to location j (where we will introduce variables only if there is a route from i to j in the network above). For locations 1,2,3 and 6,7 we introduce constraints using the outgoing and incoming routes just as before:

$$
\begin{aligned}
x_{1,4} + x_{1,6} &\leq 4000 \\
x_{2,4} + x_{2,5} &\leq 2100 \\
x_{3,5} &\leq 3000 \\
x_{1,6} + x_{4,6} + x_{5,6} &\geq 3100 \\
x_{4,7} + x_{5,7} &\geq 6000
\end{aligned}
$$

For locations 4 and 5, we need to think about how to set the supply and demand. How much is available to ship out of these locations? This can be at most the total that we ship *in* from locations 1,2,3. Thus we get constraints:

$$x_{4,6} + x_{4,7} \leq x_{1,4} + x_{2,4}$$
$$x_{5,6} + x_{5,7} \leq x_{2,5} + x_{3,5}$$

We now need to bound the capacity on each route as given in Table 2. For each i, j that has an entry $c_{i,j}$ in the table, we simply add a constraint $x_{i,j} \leq c_{i,j}$:

$$x_{1,4} \leq 2100$$
$$x_{1,6} \leq 3000$$
$$x_{2,4} \leq 1800$$
$$x_{2,5} \leq 2400$$
$$x_{3,5} \leq 5100$$
$$x_{4,6} \leq 1700$$
$$x_{4,7} \leq 4200$$
$$x_{5,6} \leq 1900$$
$$x_{5,7} \leq 5300$$

All of the inequalities we have derived so far give us the constraints for the program. The objective is to minimise the cost. Similarly to the transportation problems we have seen already, we get this by multiplying each entry in Table 1 by the corresponding decision variable and adding up the terms we get:

$$20x_{1,4} + 40x_{1,6} + 35x_{2,4} + 40x_{2,5} + 42x_{3,5} + 13x_{4,6} + 62x_{4,7} + 58x_{5,6} + 19x_{5,7}$$

This will be the objective for our minimization program.