RELATIVITY – MTH6132

PROBLEM SET 3

1. Which of the following expressions have a meaning according to the index conventions discussed in class?

$$
\begin{aligned}
(A^i B_i) C_j &= (C_j D^j) B_i, \\
(A_k B^k) C_j &= (A_m B^m) D_j \\
A_i B_j C_k D^k E_f &= M_i N_j P_f Q_k, \\
A_i B_j &= A_j B_i \\
A_m &= \frac{D_k B_m}{\sqrt{C^k B_k}}
\end{aligned}
$$

2. If lower case Latin indices take values 1 and 2, write down all the components of the following quantities in full

$$G_{ij}, \qquad A^i B_i, \qquad \Gamma^i{}_{jk}, \qquad \Gamma^i{}_{ij}, \qquad R^i{}_{jkl}, \qquad R^i{}_i$$

Hint: For example, if I had asked you to write down the components of A^i, the answer would be A^1 and A^2.

3. Let $\bar{A}$ and $\bar{B}$ denote two arbitrary 4-vectors in Minkowski spacetime. What are the conditions for $\bar{A}$ to be timelike, spacelike and null? Show that if $\bar{A}$ is timelike and $\bar{B}$ is null, then they cannot be orthogonal.

4. Consider the 4-vectors $\bar{A} = (A^0, 0, 2, 0)$ and $\bar{B} = (3, 0, B^2, 0)$, where the components A^0 and B^2 are real constants. Assuming that $\bar{A}$ is a unit spacelike vector, find A^0. Hence find B^2 if $\bar{A}$ and $\bar{B}$ are orthogonal.

5. Let $\bar{A}$ and $\bar{B}$ denote two arbitrary 4-vectors in Minkowski spacetime.

(a) Define what is meant by the scalar product $\bar{A} \cdot \bar{B}$. What does it mean to say $|\bar{A}|^2$ is an invariant?

(b) Using the fact that $|\bar{A}|^2$, $|\bar{B}|^2$ and $|\bar{A} + \bar{B}|^2$ are invariants, show that the scalar product $\bar{A} \cdot \bar{B}$ is also an invariant.

(c) Show that the sum of any two orthogonal spacelike vectors is also spacelike.

6. Consider two timelike 4-vectors, $\bar{A} = (A^0, A^1, 0, 0)$ and $\bar{B} = (B^0, B^1, 0, 0)$, where the components A^0, A^1, B^0, B^1 are all positive quantities. Show that the sum of the 4-vectors $\bar{A}$ and $\bar{B}$ can never be null.

7. Consider two inertial frames, F and F', in a standard configuration. Consider a 4-vector whose components in frame F are given by $\bar{A} = (A^0, A^1, A^2, A^3)$. Write down the norm of this vector and show that it remains invariant as we go to the frame F'. **Hint:** write down $\bar{A}' = (A'^0, A'^1, A'^2, A'^3)$ and use the transformation formula between the components of $\bar{A}$ and $\bar{A}'$.

8. The following calculation will be very useful in a few weeks. Starting from the Euclidean line element in Cartesian coordinates

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2,$$

show that in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, the line element is written as

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2.$$

Hint: In the notation from lecture, $(x^1, x^2) = (x, y)$ and $(x'^1, x'^2) = (r, \theta)$.

9. Consider the Minkowski metric written in spherical coordinates,

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,.$$

In addition, consider an anti-symmetric $(0,2)$ tensor F_{ab} (i.e., $F_{ab} = F_{[ab]}$) whose only non-vanishing components are $F_{tr} = -F_{rt} = -\frac{Q}{r^2}$, where Q is a positive constant. F_{ab} corresponds to the electromagnetic field created by a point charge Q at the origin $(r = 0)$ of space.

(a) Compute the components of the energy-momentum tensor T_{ab} for F_{ab}:

$$T_{ab} = F_{ac}\,F_b{}^c - \frac{1}{4}\,\eta_{ab}\,F_{cd}F^{cd}\,.$$

(b) Compute the trace of the energy momentum tensor, $T^a{}_a$.

10. Consider a covariant vector A_i.

(a) Write down the transformation law for A_i under a Lorentz transformation.

(b) Consider the object $F_{ij} = \partial_i A_j - \partial_j A_i$, where $\partial_i \equiv \frac{\partial}{\partial x^i}$. Compute the transformation of F_{ij} under a Lorentz transformation. Is F_{ij} a tensor?

Further Exploration. The relativistic Doppler effect plays a central role in astrophysics and cosmology. To gain additional perspective on the calculations performed in the lecture notes, we now learn about the classical Doppler effect. You have likely heard the pitch of an ambulance siren changing as it passes you on Mile End Road.

As the ambulance approaches, the pitch is high and as it recedes, the pitch is low. The classical Doppler equations

$$f_r' = \frac{f_0}{1 - u/v}, \qquad f_s' = f_0 \left(1 + \frac{u}{v}\right),$$

where v is the speed of sound, u is the relative speed of approach of the source or receiver, f_0 is the frequency of the source, f' is the frequency at the receiver. The first equation applies if the receiver is stationary and the source approaches the receiver; the second equation applies if the source is stationary and the receiver approaches the source.

- Look up a derivation of these equations in any standard physics text.
- Note the apparent lack of symmetry between the two equations. Do these equations apply for electromagnetic waves (as opposed to sound waves, such as the ambulance siren)? **Hint:** Assume the equations *were* valid for electromagnetic waves and argue that you would then be able to determine an "absolute rest frame" (for ether, e.g.).
- Think about how you might make a relativistic correction to these expressions, making them consistent with the framework of Special Relativity.
- See Section 3.10 in R. D'Inverno's book *Introducing Einstein's Relativity*, for a nice discussion on the (relativistic) Doppler effect using time dilation. Your Lecture Notes also contain a small subsection on the topic!