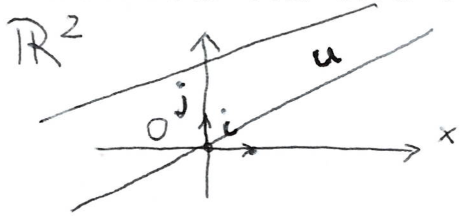
Lines through the origin and example of sub-vector spaces

Let us consider the equation of the line l passing through the origin O and defined via the vector $\mathbf{u}$, i.e., $\underline{\mathbf{r} = \lambda \mathbf{u}}$, for $\lambda \in \mathbb{R}$. This gives



such that

$$\underline{V = \{\lambda \mathbf{u} : \lambda \in \mathbb{R}\}.}$$

Proposition 4.2.1

For all $v, v_1, v_2 \in V$ and all $\alpha \in \mathbb{R}$,

- $\forall\ v_1, v_2 \in V$ we have that $v_1 + v_2 \in V$, — V is closed with respect to the addition
- $\forall\ v \in V$, $\forall\ \alpha \in \mathbb{R}$ " " " $\alpha v \in V$. — V is closed with respect to the multiplication by scalars

Proof: Let $v_1, v_2 \in V$. By definition of V, $\exists\ \lambda_1, \lambda_2 \in \mathbb{R}$:

$$v_1 = \lambda_1 u$$
$$v_2 = \lambda_2 u.$$

Hence,

$$v_1 + v_2 = \lambda_1 u + \lambda_2 u \overset{\text{distributive property}}{=} (\lambda_1 + \lambda_2) u \Longrightarrow v_1 + v_2 \in V$$

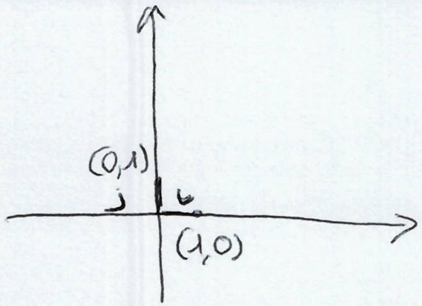
Let $v \in V$ and let $\alpha \in \mathbb{R}$. Hence, $\exists\ \lambda \in \mathbb{R}$: $v = \lambda u$.
It follows that $\alpha v = \alpha(\lambda u) = (\alpha\lambda) u \Longrightarrow \alpha v \in V$ $\square$

Proposition 4.2.2

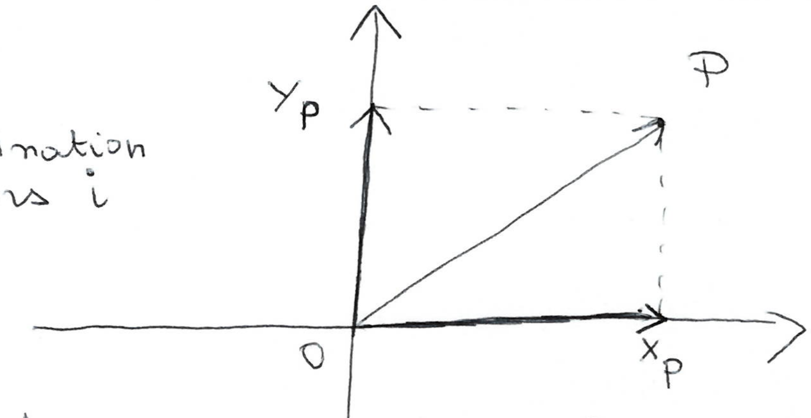
Let **i** and **j** be the standard vectors in $\mathbb{R}^2$. The set,

$$V = \{x\mathbf{i} + y\mathbf{j} : x, y \in \mathbb{R}\},$$

is a sub-vector space of $\mathbb{R}^2$. $V = \mathbb{R}^2$



$x\mathbf{i} + y\mathbf{j}$
linear combination of the vectors $\mathbf{i}$ and $\mathbf{j}$

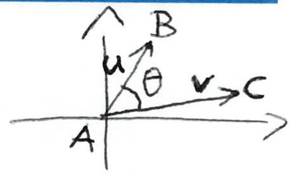


$\overline{OP} = x_P\,\mathbf{i} + y_P\,\mathbf{j}$

Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors (in $\mathbb{R}^n$)

$$\mathrm{Span}(\mathbf{u}, \mathbf{v}) = \{\alpha\mathbf{u} + \beta\mathbf{v} : \alpha, \beta \in \mathbb{R}\}.$$

Scalar product



If **u** and **v** are non-zero vectors with $\overrightarrow{AB}$ representing **u** and $\overrightarrow{AC}$ representing **v**, we define the *angle between* **u** *and* **v** to be the angle θ (in radians) between the line segments $\overrightarrow{AB}$ and $\overrightarrow{AC}$ with $0 \leq \theta \leq \pi$.

Definition 5.1.1

The *scalar product* of **u** and **v** is denoted by $\mathbf{u} \cdot \mathbf{v}$ and defined by

$$\mathbf{u} \cdot \mathbf{v} = \begin{cases} |\mathbf{u}||\mathbf{v}| \cos\theta & \text{if } \mathbf{u} \neq \mathbf{0}, \mathbf{v} \neq \mathbf{0} \\ 0 & \text{if } \mathbf{u} = \mathbf{0} \text{ or } \mathbf{v} = \mathbf{0} \end{cases}$$

where θ is the angle between **u** and **v**.

Definition 5.1.2

We say that **u** and **v** are *orthogonal* if $\mathbf{u} \cdot \mathbf{v} = 0$.

$|u| \neq 0$
$|v| \neq 0$ $\quad |u||v| \underline{\cos\theta} = 0 \quad \cos\theta = 0$

Theorem 5.1.3

If $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$. Then

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

$\mathbb{R}^3$

$\mathbb{R}^n$ $\quad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \quad v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$

$u \cdot v = \sum_{i=1}^{n} u_i v_i$

<u>Proof</u> If $u = \mathbf{0}$ or $v = \mathbf{0}$ then $u \cdot v = 0$

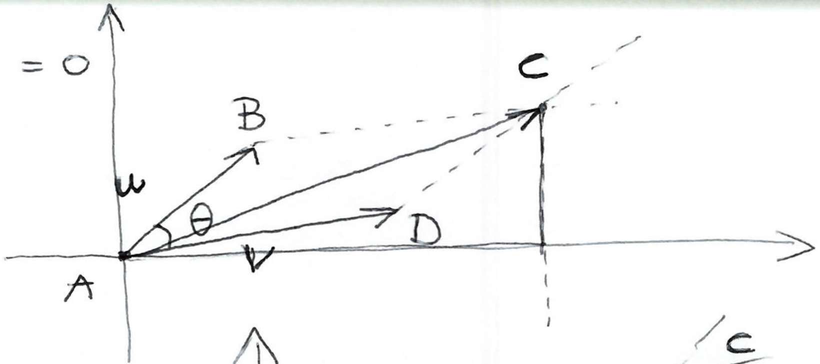
Assume that $u \neq 0$ and $v \neq 0$

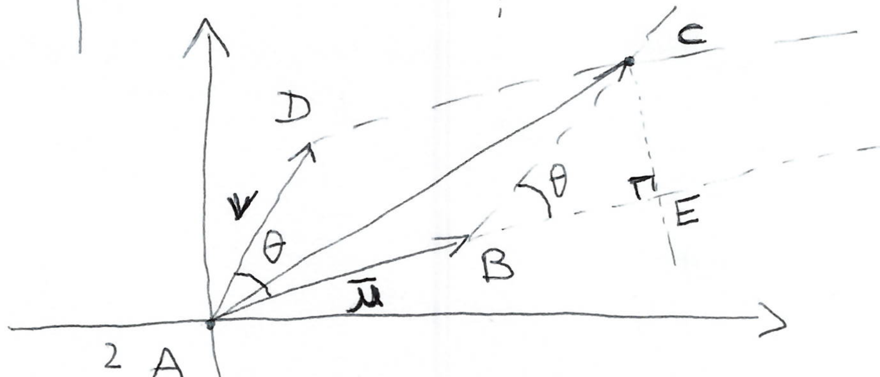
$$\boxed{|u+v|^2} = |\overrightarrow{AE}|^2 + |\overrightarrow{EC}|^2$$

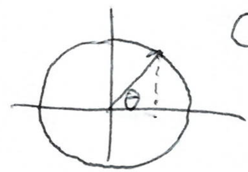
$$= (|u| + |v|\cos\theta)^2 + (|v|\sin\theta)^2$$

$$= |u|^2 + |v|^2\cos^2\theta + 2|u||v|\cos\theta + |v|^2\sin^2\theta$$

$$= \boxed{|u|^2 + |v|^2 + 2u \cdot v}$$







Circle with radius 1

$$|u+v|^2 = \left| \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{pmatrix} \right|^2 = (u_1+v_1)^2 + (u_2+v_2)^2 + (u_3+v_3)^2$$

$$= u_1^2 + v_1^2 + 2u_1v_1 + u_2^2 + v_2^2 + 2u_2v_2$$

$$+ u_3^2 + v_3^2 + 2u_3v_3$$

$$= |u|^2 + |v|^2 + 2(u_1v_1 + u_2v_2 + u_3v_3)$$

$$\cancel{|u|^2} + \cancel{|v|^2} + \cancel{2}\, u \cdot v = \cancel{|u|^2} + \cancel{|v|^2} + \cancel{2} \left(u_1 v_1 + u_2 v_2 + u_3 v_3 \right)$$

$$\implies \qquad u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$$

In $\mathbb{R}^n$ $\qquad u \cdot v = \sum_{i=1}^{n} u_i v_i$

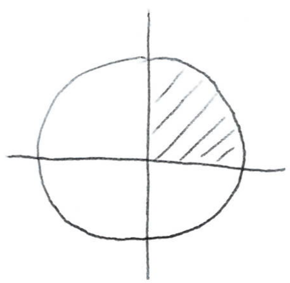
Remark 5.1.4

$u \neq 0$

$v \neq 0$

$$\cos\theta = \frac{u \cdot v}{|u|\,|v|}$$

$$\cos\theta = \frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{\sqrt{u_1^2 + u_2^2 + u_3^2}\,\sqrt{v_1^2 + v_2^2 + v_3^2}}$$



Proposition 5.1.5

For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and $\alpha \in \mathbb{R}$ we have

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$,
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$,
3. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$,
4. $(\alpha\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (\alpha\mathbf{v}) = \alpha(\mathbf{u} \cdot \mathbf{v})$.

$$u \cdot v = \sum_{i=1}^{3} u_i v_i$$

Cauchy-Schwarz Inequality

Let **u** and **v** be two vectors in $\mathbb{R}^3$. The following inequality holds:

$$|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}||\mathbf{v}|.$$

$|u \cdot v| \leq |u| \cdot |v|$

<u>Proof</u> If $u = 0$ or $v = 0$ $\quad u \cdot v = 0$, $|u||v| = 0$ $\quad 0 = 0$ ✓

Assume $u \neq 0$ and $v \neq 0$.

It is not restrictive to assume that $|u| = |v| = 1$.

$x \in \mathbb{R}$

$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

$$|u+v|^2 = (u+v)\cdot(u+v)$$

$$= u \cdot u + \underline{u \cdot v} + \underline{v \cdot u} + v \cdot v$$

$\longrightarrow 0 \leq |u+v|^2 = |u|^2 + 2\, u \cdot v + |v|^2 = 2(1 + u \cdot v)$

$\longrightarrow 0 \leq |u-v|^2 = |u|^2 - 2\, u \cdot v + |v|^2 = 2(1 - u \cdot v)$

$u \cdot u = |u||u| \cos(0)$

$u \cdot u = |u|^2$

$\boxed{1 + u \cdot v \geq 0}$ $\qquad 1 - u \cdot v \geq 0$

$-1 + u \cdot v \leq 0$

$\Longrightarrow \quad u \cdot v \geq -1 \qquad u \cdot v \leq 1 \iff \boxed{-1 \leq u \cdot v \leq 1}$

$\iff |u \cdot v| \leq 1$