Vectors & Matrices

Solutions to Problem Sheet 1

1. By definition, $\mathbf{i} = \overrightarrow{OP_1}$, $\mathbf{j} = \overrightarrow{OP_2}$, $\mathbf{k} = \overrightarrow{OP_3}$, where $P_1 = (1,0,0), P_2 = (0,1,0), P_3 = (0,0,1)$.

   If we let $Q = (1,1,0)$, then by the definition of vector addition, we have:

   $$\mathbf{i} + \mathbf{j} = \overrightarrow{OP_1} + \overrightarrow{OP_2} = \overrightarrow{OQ} \, .$$

   We can sum this derived vector with the remaining $\mathbf{k}$ to get:

   $$\overrightarrow{OQ} + \mathbf{k} = \overrightarrow{OQ} + \overrightarrow{OP_3} = \overrightarrow{OR} \, ,$$

   where $R = (1,1,1)$. But it's clear that the points A and R are identical, and so:

   $$\overrightarrow{OA} = \overrightarrow{OR} = \mathbf{i} + \mathbf{j} + \mathbf{k} \, .$$

   Per the lecture notes, the column vector

   $$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

   is simply an alternative representation of the vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, and so

   $$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k} \, .$$

   By Corollary 3.1.5 in the lecture notes, $\overrightarrow{OB} + \overrightarrow{BA} = \overrightarrow{OA}$, and so these two vectors are equivalent.

   By Proposition 3.1.6 in the notes, we find:

   $$\overrightarrow{AB} = (2-1)\mathbf{i} + (2-1)\mathbf{j} + (2-1)\mathbf{k} = \mathbf{i} + \mathbf{j} + \mathbf{k} = \overrightarrow{OA} \, ,$$

   hence $\overrightarrow{AB}$ is equivalent to $\overrightarrow{OA}$.

   The vector $\overrightarrow{OB}$ is not equivalent to $\overrightarrow{OA}$. This can be determined by computing the lengths of each vector:

$$|\overrightarrow{OA}| = |\mathbf{i}+\mathbf{j}+\mathbf{k}| = (1^2+1^2+1^2)^{\frac{1}{2}} = \sqrt{3}\ ,$$
$$|\overrightarrow{OB}| = |2\mathbf{i}+2\mathbf{j}+2\mathbf{k}| = (2^2+2^2+2^2)^{\frac{1}{2}} = 2\sqrt{3}\ ,$$

Clearly, $\overrightarrow{OB}$ has twice the length of $\overrightarrow{OA}$, and so the two vectors cannot be equivalent.

2. By two applications of Corollary 3.1.5, we have:

$$\begin{aligned}\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD} &= (\overrightarrow{AB}+\overrightarrow{BC})+\overrightarrow{BC}\\ &= \overrightarrow{AC}+\overrightarrow{CD}\\ &= \overrightarrow{AD}\ .\end{aligned}$$

Therefore, taking the sums of the components of each of the three vectors on the left-hand side of the above equality:

$$\overrightarrow{AD} = (1+3-1)\mathbf{i} + (-2+1+5)\mathbf{j} + (-5+4+2)\mathbf{k} = 3\mathbf{i}+4\mathbf{j}+\mathbf{k}\ .$$

Since the components of a vector in the standard basis **i**, **j**, **k** are equal to the end point of its position vector, we see that $E = (3, 4, 1)$.

3. (i) Let $A = (x_A, y_A, z_A)$. We also define the point $B = (x_B, y_B, z_B)$ to be the point such that $\overrightarrow{AO} = \overrightarrow{OB}$. (This point can be obtained by simply translating the vector $\overrightarrow{AO}$ so that its starting point becomes the origin, and then finding the new end point of the vector).

   By Corollary 3.1.5, we know that $\overrightarrow{OA}+\overrightarrow{AO} = \overrightarrow{OA}+\overrightarrow{OB} = \overrightarrow{OO}$.

   The end point of the vector on the right-hand side of the above equation is the origin, $O = (0, 0, 0)$. Since $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are both position vectors, we know that their sum must give a position vector with end point equal the sums of A and B.

   Hence, $x_A + x_B = 0$, $y_A + y + B = 0$ and $z_A + z_B = 0$, and so $B = (-x_A, -y_A, -z_A) = -1 \cdot (x_A, y_A, z_A)$.

   Since $\overrightarrow{OB}$ is a position vector, by the definition of scalar multiplication of vectors, $\overrightarrow{OB} = -1 \cdot \overrightarrow{OA} = -\overrightarrow{OA}$.

   (ii) By Corollary 3.1.5, $\overrightarrow{AB} = \overrightarrow{AO}+\overrightarrow{OB}$.

   We know from part (i) that $\overrightarrow{AO} = -\overrightarrow{OA}$, and so $\overrightarrow{AB} = -\overrightarrow{OA}+\overrightarrow{OB}$.

Negation is simply multiplication by a factor of -1, and so we can use the distributivity of scalar multiples over vector sums (property (vii) in Proposition 3.2.1) to get:

$$\begin{aligned}-\overrightarrow{AB} &= -1 \cdot \overrightarrow{AB} \\ &= -1 \cdot (-\overrightarrow{OA} + \overrightarrow{OB}) \\ &= -1 \cdot (-1 \cdot \overrightarrow{OA} + \overrightarrow{OB}) \\ &= (-1)(-1)\overrightarrow{OA} + (-1)\overrightarrow{OB}) \\ &= \overrightarrow{OA} - \overrightarrow{OB}\end{aligned}$$

Proposition 3.2.1 also gives us the commutativity of vector addition in property (i), we can use this to show $-\overrightarrow{AB} = \overrightarrow{OA} - \overrightarrow{OB} = -\overrightarrow{OB} + \overrightarrow{OA}$.

We again use the fact that $-\overrightarrow{OB} = \overrightarrow{BO}$, combined with Corollary 3.1.5, to get:

$$-\overrightarrow{AB} = -\overrightarrow{OB} + \overrightarrow{OA} = \overrightarrow{BO} + \overrightarrow{OA} = \overrightarrow{BA} \, .$$

4. (i) Per the previous question, $\overrightarrow{QO} = -\overrightarrow{OQ} = -\mathbf{q}$.

(ii) Let $\mathbf{r} = \overrightarrow{OR}$ be the position vector of the point R. By definition Corollary 3.1.5, we have:

$$\mathbf{q} = \overrightarrow{OQ} = \overrightarrow{OR} + \overrightarrow{RQ} \, ,$$

but since $\overrightarrow{RQ} = \overrightarrow{OP}$:

$$\mathbf{q} = \overrightarrow{OR} + \overrightarrow{OP} = \mathbf{r} + \mathbf{p} \, .$$

Hence $\overrightarrow{OR} = \mathbf{r} = \mathbf{q} - \mathbf{p}$.

(iii) Again, by Corollary 3.1.5,

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = -\overrightarrow{OP} + \overrightarrow{OQ} = -\mathbf{p} + \mathbf{q} = \mathbf{q} - \mathbf{p} \, .$$

(iv) Since $OPQR$ is a parallelogram, we know that $\overrightarrow{RQ} = \overrightarrow{OP} = \mathbf{p}$, hence $\overrightarrow{QR} = -\overrightarrow{RQ} = -\mathbf{p}$.

(v) By Corollary 3.15:

$$\overrightarrow{RP} = \overrightarrow{RO} + \overrightarrow{OP} = -\overrightarrow{OR} + \overrightarrow{OP} \, .$$

Again, letting $\mathbf{r} = \overrightarrow{OR}$ gives us $\overrightarrow{RP} = -\mathbf{r} + \mathbf{p}$.

But we know from part (ii) that $\mathbf{r} = \mathbf{q} - \mathbf{p}$, and so:

$$\overrightarrow{RP} = -(\mathbf{q} - \mathbf{p}) + \mathbf{p} = -\mathbf{q} + \mathbf{p} + \mathbf{p} = 2\mathbf{p} - \mathbf{q}.$$

5. Suppose we write

$$\mathbf{v} = \sum_{i=1}^{n} v_i \mathbf{e_i} \ ,$$

where e_i are the standard basis vectors over $\mathbf{R}^n$.

By the distributivity of scalar multiples over vector sums (property (vii) of Proposition 3.2.1), we have:

$$\lambda \mathbf{v} = \sum_{i=1}^{n} \lambda v_i \mathbf{e_i} \ .$$

Hence, by the definition of the length operator:

$$|\lambda \mathbf{v}| = \sqrt{\sum_{i=1}^{n} (\lambda v_i)^2} = \sqrt{\sum_{i=1}^{n} \lambda^2 \cdot v_i^2} = \sqrt{\lambda^2 \cdot \sum_{i=1}^{n} v_i^2} \ ,$$

where the last equality was obtained by factoring out the common factor of λ^2 present in every term of the sum.

We can now split the square-root between these factors to obtain:

$$|\lambda \mathbf{v}| = \sqrt{\lambda^2 \cdot \sum_{i=1}^{n} v_i^2} = \sqrt{\lambda^2} \sqrt{\sum_{i=1}^{n} v_i^2} = |\lambda||\mathbf{v}| \ .$$

Geometrically, this tells us that the effect of multiplying a vector by a scalar λ is to extend the length of the vector by that same proportion.