**MTH5114** **Linear Programming and Games, Spring 2023**

**Week 1 Seminar Questions** **Viresh Patel**

---

**Sample Question (From 2019 Exam):** Rewrite the following linear program in *standard inequality form*:

$$\begin{aligned}
\text{minimize} \quad & -3x_1 + 6x_2 + 9x_3 + 12x_4 \\
\text{subject to} \quad & x_1 + x_2 - x_4 \geq 3, \\
& x_1 + x_2 - 3x_4 \leq 5, \\
& 3x_1 + 2x_2 + x_3 = 9, \\
& x_1, x_2 \geq 0, \\
& x_4 \leq 0, \\
& x_3 \text{ unrestricted}
\end{aligned}$$

If $(x_1, x_2, x_3, x_4) = (a, b, c, d)$ is an optimal solution for the linear program above, write down an optimal solution for the linear program in standard form that you have found in the first part of the question. (You may have to consider some cases.)

**Solution:**

First we fix the variables by taking $x_4 = -\bar{x}_4$ and $x_3 = x_3^+ - x_3^-$. This gives

$$\begin{aligned}
\text{minimize} \quad & -3x_1 + 6x_2 + 9x_3 + 12x_4 \\
\text{subject to} \quad & x_1 + x_2 + \bar{x}_4 \geq 3, \\
& x_1 + x_2 + 3\bar{x}_4 \leq 5, \\
& 3x_1 + 2x_2 + x_3^+ - x_3^- = 9, \\
& x_1, x_2, x_3^+, x_3^-, \bar{x}_4 \geq 0
\end{aligned}$$

Next we fix the objective as well as the first and third constraints (the second constraint is already in standard form). This gives:

$$\begin{aligned}
\text{maximize} \quad & 3x_1 - 6x_2 - 9x_3^+ + 9x_3^- + 12\bar{x}_4 \\
\text{subject to} \quad & -x_1 - x_2 - \bar{x}_4 \leq -3, \\
& x_1 + x_2 + 3\bar{x}_4 \leq 5, \\
& 3x_1 + 2x_2 + x_3^+ - x_3^- \leq 9, \\
& -3x_1 - 2x_2 - x_3^+ + x_3^- \leq -9, \\
& x_1, x_2, x_3^+, x_3^-, \bar{x}_4 \geq 0
\end{aligned}$$

(Note that the next part of the question was not in the exam.) Suppose $(x_1, x_2, x_3, x_4) = (a, b, c, d)$ is an optimal solution for the given linear program for some numbers $a, b, c, d$. If

$c > 0$ then $(x_1, x_2, x_3^+, x_3^-, \bar{x}_4) = (a, b, c, 0, -d)$ is an optimal solution for the linear program in standard form found above. If $c < 0$ then $(x_1, x_2, x_3^+, x_3^-, \bar{x}_4) = (a, b, 0, -c, -d)$ is an optimal solution. [Extension: can you explain why this is an optimal solution.]

---

**Questions for discussion** Questions 1 and 2 are for revision of linear algebra in preparation for some of the proofs in the module (around week 4). Questions 3 and 4 test how well you have understood the definition of optimal solution.

1. (a) Show that the vector $\mathbf{y} = A\mathbf{x}$ is a linear combination of the columns of $A$.

   (b) Show that if a square matrix $A$ is invertible then its columns must be linearly independent.

   **Solution:**

   (a) Suppose that $A$ is an $m \times n$ matrix. Then, writing out the matrix multiplication we find that $\mathbf{y}$ is simply:

$$\begin{pmatrix} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \end{pmatrix} = x_1 \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m,1} \end{pmatrix} + x_2 \begin{pmatrix} a_{1,2} \\ a_{2,2} \\ \vdots \\ a_{m,2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{m,n} \end{pmatrix}$$

   This is a linear combination of the columns of $A$, where the $i$th column is multiplied by $x_i$.

   (b) Suppose that the columns of $A$ are *not* linearly independent. Then, there is some non-trivial linear combination of them that is equal to $\mathbf{0}$. In other words, we can find $\mathbf{x} \neq 0$ so that $A\mathbf{x} = \mathbf{0}$. But, we also know that $A\mathbf{0} = \mathbf{0}$. Thus, $A$ can have no inverse, since it maps 2 distinct vectors to the same vector $\mathbf{0}$. In particular, notice that we can't choose a single value for $A^{-1}\mathbf{0}$, since it should be both $\mathbf{x}$ and $\mathbf{0}$.

2. Determine whether or not each of the following vectors is linearly independent:

   (a) $\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 8 \\ 8 \end{pmatrix}$

   (c) Any set of $m$ vectors of length $n$ where $m > n$.

   **Solution:**

(a) These vectors are linearly independent. To show this, suppose that some linear combination of them equals $\mathbf{0}$. Then, we must have values $a_1, a_2, a_3$ such that:

$$\begin{aligned} a_1 &= 0 \\ 3a_3 + a_2 + a_3 &= 0 \\ 4a_2 + a_3 &= 0 \end{aligned}$$

The unique solution to these equations is $a_1 = a_2 = a_3 = 0$. Thus, the only linear combination of the vectors that equals $\mathbf{0}$ is the trivial, "all zero" combination.

(b) These vectors are linearly dependent, since taking 1 times the first plus 1 times the second minus 1/2 times the third gives $\mathbf{0}$. You can find this either by trial-and-error or by writing out the equations as in the previous question and finding that they have many non-trivial solutions.

(c) If we put the vectors as columns of a matrix $A$ then $A$ has $n$ rows and $m$ columns with $n < m$. It can have at most $n$ linearly independent rows, which means its rank is at most $n$. Since row rank and column rank are equal, then the column rank of $A$ is also at most $n < m$. Since $A$ has $m$ columns but has column rank strictly less than $m$, then these $m$ columns must be linearly dependent.

3. When defining linear programs, we have assumed that the objective function is a linear combination $\mathbf{c}^\mathsf{T}\mathbf{x}$, and so the solution $\mathbf{x} = \mathbf{0}$ always has objective value 0. What if we wanted to maximise a quantity such as: $3x_1 + 2x_2 - 4x_3 + 10$? More generally, suppose we want to solve a problem of the form:

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^\mathsf{T}\mathbf{x} + k \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{1}$$

where $k$ is a constant.

(a) Find a linear program whose optimal solution $\mathbf{x}$ is the same as the optimal solution for the above problem.

(b) Prove that your program has this property (that is, that its optimal solution is the same as the optimal solution for the given mathematical program).

**Solution:**

(a) Intuitively, the constant $k$ will be added to the value of *every* solution, so it won't change which solution is the largest. We can then just solve the linear program:

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

to find the optimal $\mathbf{x}$ and remember that its actual value is $\mathbf{c}^\mathsf{T}\mathbf{x} + k$.

(b) Suppose that $\mathbf{x}$ is an optimal solution to the given problem. This happens if and only if $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{c}^\mathsf{T}\mathbf{x} + k \geq \mathbf{c}^\mathsf{T}\mathbf{y} + k$ for any other $\mathbf{y}$ with $A\mathbf{y} \leq \mathbf{b}$ and $\mathbf{y} \geq \mathbf{0}$. But, for any such $\mathbf{y}$, $\mathbf{c}^\mathsf{T}\mathbf{x} + k \geq \mathbf{c}^\mathsf{T}\mathbf{y} + k$ if and only if $\mathbf{c}^\mathsf{T}\mathbf{x} \geq \mathbf{c}^\mathsf{T}\mathbf{y}$. Thus, $\mathbf{x}$ is an optimal solution to the given program if and only if $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{c}^\mathsf{T}\mathbf{x} \leq \mathbf{c}^\mathsf{T}\mathbf{y}$ for any $\mathbf{y}$ with $A\mathbf{y} \leq \mathbf{b}$ and $\mathbf{y} \geq \mathbf{0}$, which in turn is equivalent to saying that $\mathbf{x}$ is an optimal solution to our program from part (a).

4. Consider an arbitrary linear program in standard inequality form. Determine which of the following transformations will change the optimal solution to this program:

(a) Multiplying the objective by a non-negative constant $k$.

(b) Multiplying the matrix $A$ by a non-negative constant $k$.

(c) Multiplying the matrix $\mathbf{b}$ by a non-negative constant $k$.

(d) Multiplying both $A$ and $\mathbf{b}$ by a non-negative constant $k$.

**Solution:**

(a) This will not change the optimal solution, since $\mathbf{c}^\mathsf{T}\mathbf{x} \geq \mathbf{c}^\mathsf{T}\mathbf{y}$ if and only if $k\mathbf{c}^\mathsf{T}\mathbf{x} \geq k\mathbf{c}^\mathsf{T}\mathbf{y}$. That is, scaling the objective by $k$ preserves the relative ordering of feasible solutions by objective value. Note that it *will* change the *objective value* of the optimal solution—specifically, it will scale this value by $k$. However, the choice of $\mathbf{x}$ that maximises the objective will stay the same for both programs.

(b) This will potentially change the optimal solution, since it changes the right-hand side of all inequalities in the program, which will change the set of values that satisfy them (i.e. the set of feasible solutions to the program). As a simple example:

$$\begin{aligned} \text{maximize} \quad & x_1 \\ \text{subject to} \quad & x_1 \leq 3, \\ & x_1 \geq 0 \end{aligned}$$

(obviously) has optimal solution $x_1 = 3$. But if we multiply the matrix $A$ (note that here $A$ contains only a single entry, which is 1) by $k$, then the program has optimal solution $x_1 = 3/k$.

(c) This will potentially change the optimal solution. Again, this is because multiplying only the left-hand side of an inequality by a constant does not in general give the same inequality. Notice that if we multiply the right-hand side of the single constraint in the previous example by $k$, then the optimal solution changes to $x_1 = 3k$.

(d) This will *not* change the optimal solution, since it effectively multiplies each inequality of the program by a non-negative constant, which results in an equivalent inequality. Thus, the resulting program will have the same set of feasible solutions and since the objective function has not changed, it will have the same optimal solution.