Chapter 4 : Equations of lines

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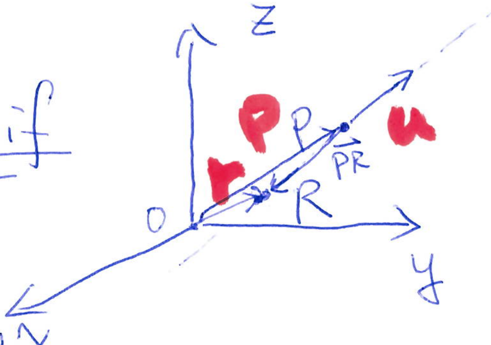
lines in $\mathbb{R}^3$. The definition in this chapter can be easily generalised to the n-dimensional context $\mathbb{R}^n$

4.1. Equations of lines

4.1.1. Parametric type equations

let l be the line through the point P in the direction of the non-zero vector $\mathbf{u}$

The point R with position vector $\mathbf{r}$ is on the line l <u>if and only if</u> the vector represented by $\overrightarrow{PR}$ is a multiple of $\mathbf{u}$.



That is, R is on l <u>if and only if</u>

$$\mathbf{r} - \mathbf{p} = \lambda \mathbf{u}$$

for some $\lambda \in \mathbb{R}$. or equivalently if and only if

$$\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR} = \overrightarrow{OR} + \overrightarrow{PO} = \overrightarrow{OR} - \overrightarrow{OP} = \mathbf{r} - \mathbf{p} = \lambda \mathbf{u}$$

Definition 4.1.1 The equation $\mathbf{r} = \mathbf{p} + \lambda \mathbf{u}$ is called the <u>vector equation</u> for the line l

Note that in this equation, $\mathbf{r}$ is a (vector) variable depending on the (real number) variable λ. $\mathbf{p}$ and $\mathbf{u}$ are constant vectors.

This equation gives a condition which $\mathbf{r}$ satisfies if and only if point R lies on the line l.

Working in coordinates let $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. $\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$ $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$. We get the point R is on the line l if and only if

$$\underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\text{position vector of point } R} = \underbrace{\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}}_{\text{position vector of point } P} + \lambda \underbrace{\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}}_{\text{non-zero vector defining the direction of line } l}$$

This is equivalent to the system of equations

$$\begin{cases} x = p_1 + \lambda u_1 \\ y = p_2 + \lambda u_2 \\ z = p_3 + \lambda u_3 \end{cases} \qquad 4.1.1$$

Definition 4.1.2 The equations 4.1.1 are called <u>the parametric equations</u> for the line l

Summary. the line ℓ can be defined by either the vector equation or the parametric equations.

$\mathbf{r}$ variable in LHS

λ . variable in RHS

x, y, z Variables in the LHS

λ in RHS

4.1.2 Cartesian equations

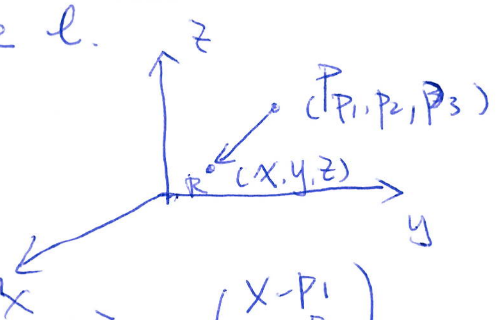
$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

- if $u_1 \neq 0,\ u_2 \neq 0,\ u_3 \neq 0$. then we can elimate the parameter λ from the parametric equations 4.1.1

$$\lambda = \frac{x - p_1}{u_1}, \quad \lambda = \frac{y - p_2}{u_2}, \quad \lambda = \frac{z - p_3}{u_3}$$

$$\Rightarrow \quad \boxed{\frac{x - p_1}{u_1} = \frac{y - p_2}{u_2} = \frac{z - p_3}{u_3}}$$

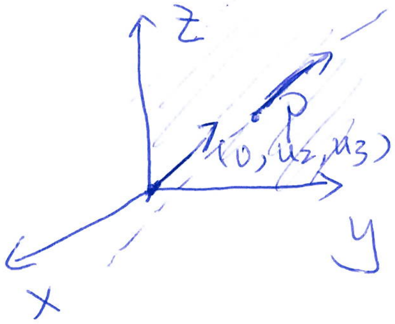
Cartesian equation for the line ℓ.



$$\vec{PR} = \begin{pmatrix} x - p_1 \\ y - p_2 \\ z - p_3 \end{pmatrix}$$

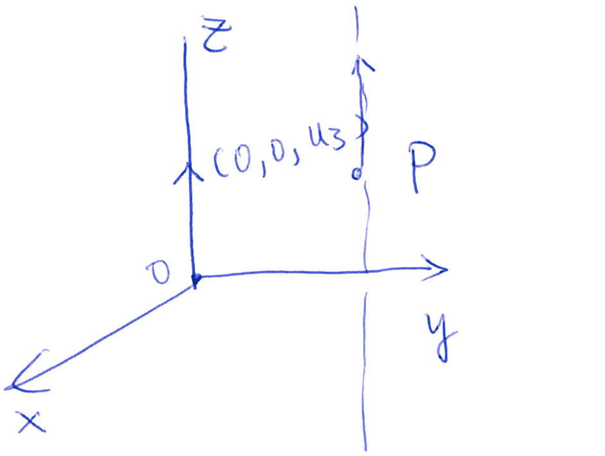
- if $u_1 = 0,\ u_2 \neq 0,\ u_3 \neq 0$ the corresponding Cartesian equation will be

$$x = p_1, \quad \frac{y - p_2}{u_2} = \frac{z - p_3}{u_3}$$



- if $u_1 = u_2 = 0$, $u_3 \neq 0$. the cartesian equations of line ℓ will be

$$x = p_1, \quad y = p_2 \qquad \text{with no constrains on } z$$



4.1.3. line described by two points.

Another natural way of describing a line is by giving two points that lie on it. If P and Q are distinct points with position vectors $\mathbf{p}$, $\mathbf{q}$ respectively. then the line containing P and Q is in direction $\mathbf{p} - \mathbf{q}$ $\quad \mathbf{u} = \mathbf{p} - \mathbf{q}$

$$\mathbf{r} = \mathbf{p} + \lambda(\mathbf{p} - \mathbf{q})$$

position vector of point R

position vector of point ~~q~~ P

position vector of point q.

which is equivalent

$$\mathbf{r} = \mathbf{q} + \lambda(\mathbf{q} - \mathbf{p}) \qquad \text{for some } \lambda \in \mathbb{R}$$

any set $\{\mathbf{r} : \mathbf{r} = \mathbf{p} + \cancel{2}\lambda\mathbf{u}\}$ 4