Vectors in the plane and space

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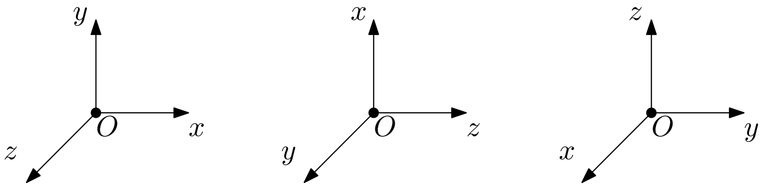
Position vectors and geometrical interpretation

Definition 3.1.1

Let A and B be two points in $\mathbb{R}^3$. The vector $\overrightarrow{AB}$ is the segment with starting point A and ending point B.

Cartesian coordinates in $\mathbb{R}^3$ and vectors



For the sake of simplicity we will fix the third system of axes in the figure above.

Let us consider the points $O = (0,0,0)$, $P_1 = (1,0,0)$, $P_2 = (0,1,0)$ and $P_3 = (0,0,1)$ and the vectors

$$\begin{aligned} &\overrightarrow{OP_1}, \qquad \overrightarrow{OP_1} = \mathbf{i}, \\ &\overrightarrow{OP_2}, \qquad \overrightarrow{OP_2} = \mathbf{j}, \\ &\overrightarrow{OP_3}, \qquad \overrightarrow{OP_3} = \mathbf{k}. \end{aligned}$$

Definition 3.1.2

Let $A = (x_A, y_A, z_A)$ be a point in $\mathbb{R}^3$. The position vector $\overrightarrow{OA}$ is defined by $x_A\mathbf{i} + y_A\mathbf{j} + z_A\mathbf{k}$ and it is geometrically represented by the segment with starting point O and ending point A.

Definition 3.1.3

Let $\mathbb{R}^n$ the set of all n-ple $z = (z_1, z_2, \cdots, z_n)$
of real numbers z_i, $i = 1, \cdots, n$. Addition and multiplication by real scalars in $\mathbb{R}^n$ are defined as follows:

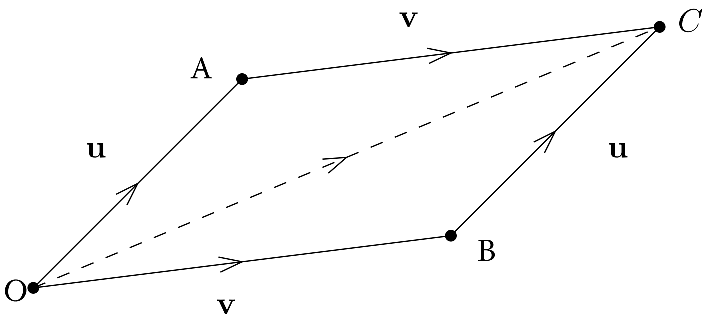
(i) for all $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$,

$$x + y = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n);$$

(ii) for all $x = (x_1, x_2, \cdots, x_n)$ and $\lambda \in \mathbb{R}$,

$$\lambda x = (\lambda x_1, \lambda x_2, \cdots, \lambda x_n)$$

The parallelogram rule



$\overrightarrow{OC}$ is the diagonal of the parallelogram with start point O. We need to prove that $\overrightarrow{OC} = \mathbf{u} + \mathbf{v}$.

Proposition 3.1.4

The sum of the vectors $\overrightarrow{OA} + \overrightarrow{OB}$ is the vector $\overrightarrow{OC}$ which represents the diagonal of the parallelogram constructed on $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

Proof

Corollary 3.1.5

Let $OBCA$ be the parallelogram above. Then,

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC},$$
$$\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC}.$$

Proposition 3.1.6

Let $A = (x_A, y_A, z_A)$ and $B = (x_B, y_B, z_B)$ be two points in $\mathbb{R}^3$. The vector $\overrightarrow{AB}$ is the sum

$$\overrightarrow{AO} + \overrightarrow{OB} = (x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k}.$$

Vectors in $\mathbb{R}^n$

Proposition 3.2.1

$\mathbb{R}^n$ is a set closed with respect to addition and scalar multiplication. In addition, the following properties hold:

(i) for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$, (addition in $\mathbb{R}^n$ is commutative),

(ii) for all $\mathbf{v}, \mathbf{w}, \mathbf{z} \in \mathbb{R}^n$, $(\mathbf{v} + \mathbf{w}) + \mathbf{z} = \mathbf{v} + (\mathbf{w} + \mathbf{z})$, (addition in $\mathbb{R}^n$ is associative),

(iii) for all $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{v} + \mathbf{0} = \mathbf{v}$, ($\mathbf{0}$ is the identity for the addition),

(iv) for all $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{v} + (-\mathbf{v}) = 0$ ($-\mathbf{v}$ is the additive inverse of $\mathbf{v}$),

(v) for all $\alpha, \beta \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^n$, $(\alpha\beta)\mathbf{v} = \alpha(\beta\mathbf{v})$, (multiplication by scalars is associative),

(vi) for all $\mathbf{v} \in \mathbb{R}^n$, $1\mathbf{v} = \mathbf{v}$, (1 is the identity for the multiplication by scalars),

(vi) for all $\alpha \in \mathbb{R}$ and $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, $\alpha(\mathbf{v}, +\mathbf{w}) = \alpha\mathbf{v} + \alpha\mathbf{w}$, (distributive property),

(vii) or all $\alpha\, \beta \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^n$, $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \alpha\mathbf{w}$, (distributive property).

Definition 3.2.3

Let $i = 1, \cdots, n$. The standard vector $\mathbf{e_i}$ is the column vector with the i-th entry equal to 1 and all the others equal to 0.

Proposition 3.2.4

Every vector $\mathbf{v}$ in $\mathbb{R}^n$ can be written as a unique linear combination of the standard vectors, i.e., there exists a unique choice of $v_i \in \mathbb{R}$, $i = 1, \cdots, n$, such that

$$\mathbf{v} = \sum_{i=1}^{n} v_i \mathbf{e_i}.$$

Definition 3.2.6

Let $\mathbf{v} = \sum_{i=1}^{n} v_i \mathbf{e_i}$ be a vector in $\mathbb{R}^n$. Its length (or norm or module) is defined as

$$|\mathbf{v}| = \sqrt{\sum_{i=1}^{n} v_i^2}.$$

Proposition 3.2.7

Let $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and let P be the point in $\mathbb{R}^3$ with coordinates (a, b, c). Then $|\mathbf{v}|$ is the length of the segment OP.