MTH5114: Linear Programming and Game Theory

Semester B 2024

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Today (first hour)

- What is the module about?
- Organisation of the module + questions

We’re Going to Learn about Computer Programming, Right?

“Programming”

- The term “Programming” often has a different meaning in Operational Research:

> “A mathematical method of solving practical problems (e.g. allocation of resources) by means of linear functions where the variables involved are subject to constraints”

- In this module, it means we will see how to:
  - Model decision-making problems mathematically
  - Find an optimal solution
- For example, a “**programme**” for production.

Mathematical Program

$$
\begin{array}{ll}
\text{minimise} & 0.25x_1 + 0.60x_2 + 0.85x_3 + 1.00x_4 + 0.80x_5 + 0.50x_6 \\
\text{subject to} & 0.1x_1 + 0.2x_2 + 0.0x_3 + 0.2x_4 + 0.2x_5 + 0.2x_6 \geq 0.8 \\
& 0.1x_1 + 0.1x_2 + 0.5x_3 + 1.2x_4 + 0.1x_5 + 0.1x_6 \geq 1.1 \\
& 1.3x_1 + 1.1x_2 + 0.1x_3 + 0.2x_4 + 0.5x_5 + 4.2x_6 \geq 13 \\
& 0.0x_1 + 0.0x_2 + 0.0x_3 + 0.0x_4 + 95.8x_5 + 28.7x_6 \geq 35 \\
& x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{array}
$$

Example 1.1. A university student is planning her daily food budget. Based on the British Nutrition Foundation's guidelines for an average female of her age she should consume the following daily amounts of vitamins:

Vitamin	mg/day
Thiamin	0.8
Riboflavin	1.1
Niacin	13
Vitamin C	35

After doing some research, she finds the following cost, calories, and vitamins (in mg) per serving of several basic foods:

Food	Cost	Thiamin	Riboflavin	Niacin	Vitamin C
Bread	£0.25	0.1	0.1	1.3	0.0
Beans	£0.60	0.2	0.1	1.1	0.0
Cheese	£0.85	0.0	0.5	0.1	0.0
Eggs	£1.00	0.2	1.2	0.2	0.0
Oranges	£0.80	0.2	0.1	0.5	95.8
Potatoes	£0.50	0.2	0.1	4.2	28.7

How can the student meet her daily requirements as cheaply as possible?

Vitamin	mg/day
Thiamin	0.8
Riboflavin	1.1
Niacin	13
Vitamin C	35

Food	Cost	Thiamin	Riboflavin	Niacin	Vitamin C
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Oranges	£0.80	0.2	0.1	0.5	95.8
Potatoes	£0.50	0.2	0.1	4.2	28.7

Variables: $x_1, x_2, x_3, x_4, x_5, x_6$

(# Daily Servings of Bread, Beans, Cheese, Eggs, Oranges, and Potatoes)

Daily Cost:

$$0.25x_1 + 0.60x_2 + 0.85x_3 + 1.00x_4 + 0.80x_5 + 0.50x_6$$

Thiamin Constraint:

$$0.1x_1 + 0.2x_2 + 0.0x_3 + 0.2x_4 + 0.2x_5 + 02.x_6 \geq 0.8$$

Mathematical Program

Goal

Objective Function

$$
\begin{aligned}
\text{minimise} \quad & 0.25x_1 + 0.60x_2 + 0.85x_3 + 1.00x_4 + 0.80x_5 + 0.50x_6 \\
\text{subject to} \quad & 0.1x_1 + 0.2x_2 + 0.0x_3 + 0.2x_4 + 0.2x_5 + 0.2x_6 \geq 0.8 \\
& 0.1x_1 + 0.1x_2 + 0.5x_3 + 1.2x_4 + 0.1x_5 + 0.1x_6 \geq 1.1 \\
& 1.3x_1 + 1.1x_2 + 0.1x_3 + 0.2x_4 + 0.5x_5 + 4.2x_6 \geq 13 \\
& 0.0x_1 + 0.0x_2 + 0.0x_3 + 0.0x_4 + 95.8x_5 + 28.7x_6 \geq 35 \\
& x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{aligned}
$$

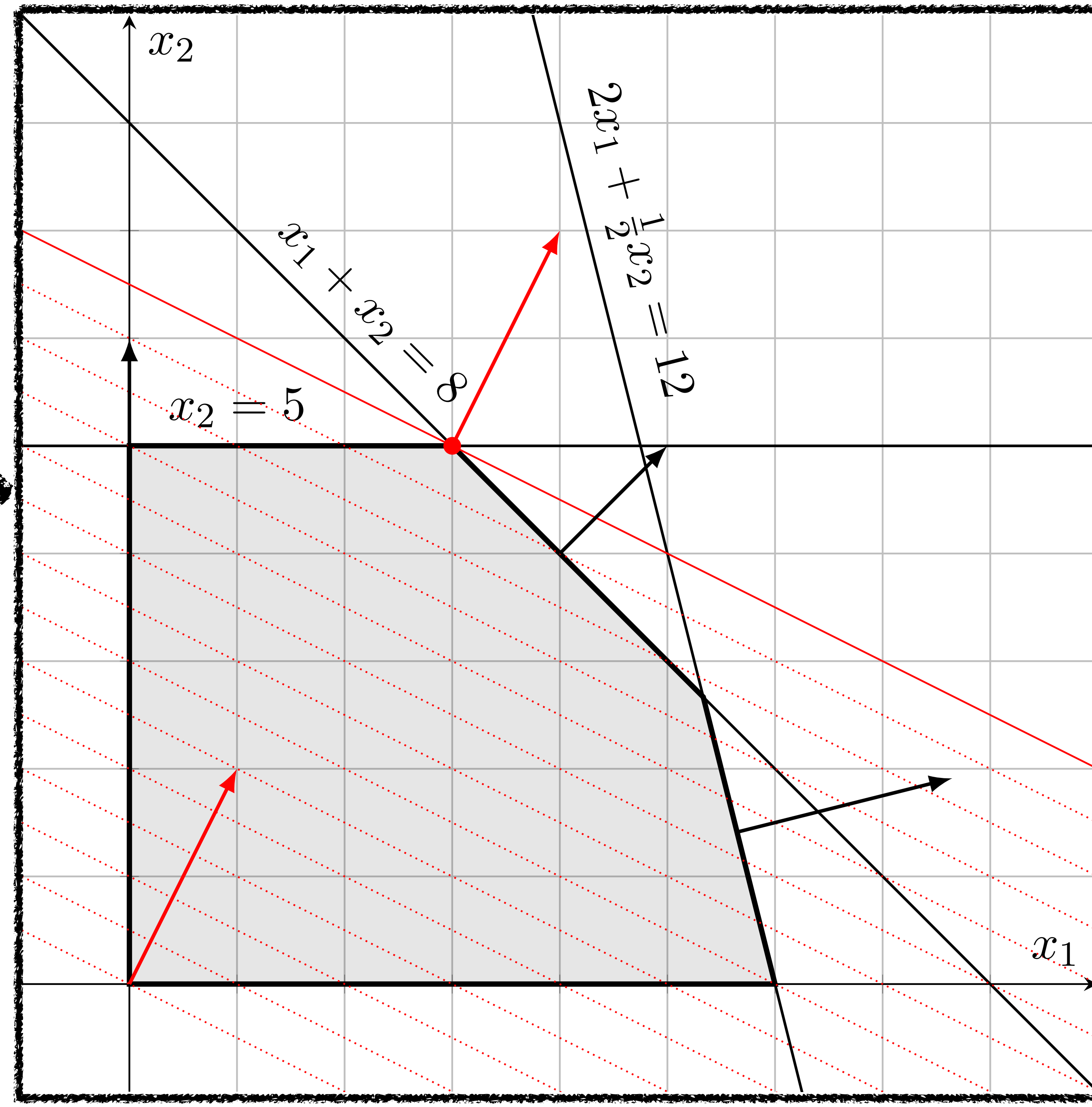
Constraints

Variables

Restrictions

Geometry

$$
\begin{aligned}
\text{maximise} \quad & x_1 + 2x_2 \\
\text{subject to} \quad & x_2 \le 5 \\
& x_1 + x_2 \le 8 \\
& 2x_1 + \frac{1}{2}x_2 \le 12 \\
& x_1, x_2 \ge 0
\end{aligned}
$$



Simplex Algorithm

- We will see that there is a **simple algorithm** that is **guaranteed** to **optimally** solve all mathematical programs of this form.
- For our diet example, the optimal solution costs: **£2.27**
- It consists of (daily):
  - 2.03 servings of bread
  - 0.54 servings of eggs
  - 2.44 servings of potatoes

Vitamin	mg/day
Thiamin	0.8
Riboflavin	1.1
Niacin	13
Vitamin C	35

Food	Cost	Thiamin	Riboflavin	Niacin	Vitamin C
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Eggs	£1.00	0.2	1.2	0.2	0.0
Oranges	£0.80	0.2	0.1	0.5	95.8
Potatoes	£0.50	0.2	0.1	4.2	28.7

Example 1.2. Suppose you run a company that sells dietary supplements. You make four kinds of tablets for, respectively, Thiamin, Riboflavin, Niacin, and Vitamin C. You decide to market them to poor university students as a cheaper way to meet their daily vitamin requirements. How should you price your supplements to maximise the revenue you can obtain for one day's worth of supplements?

Vitamin	mg/day
Thiamin	0.8
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Niacin	13
Vitamin C	35

Food	Cost	Thiamin	Riboflavin	Niacin	Vitamin C
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Variables: y_1, y_2, y_3, y_4

(Price for 1mg of Thiamin, Riboflavin, Niacin, and Vitamin C)

Daily Profit:

$$0.8y_1 + 1.1y_2 + 13.0y_3 + 35.0y_4$$

Bread Constraint:

$$0.1y_1 + 0.1y_2 + 1.3y_3 + 0.0y_4 \leq 0.25$$

Mathematical Program

$$
\begin{aligned}
\text{maximise} \quad & 0.8y_1 + 1.1y_2 + 13y_3 + 35y_4 \\
\text{subject to} \quad & 0.1y_1 + 0.1y_2 + 1.3y_3 + 0.0y_4 \le 0.25 \\
& 0.2y_1 + 0.1y_2 + 1.1y_3 + 0.0y_4 \le 0.60 \\
& 0.0y_1 + 0.5y_2 + 0.1y_3 + 0.0y_4 \le 0.85 \\
& 0.2y_1 + 1.2y_2 + 0.2y_3 + 0.0y_4 \le 1.00 \\
& 0.2y_1 + 0.1y_2 + 0.5y_3 + 95.8y_4 \le 0.80 \\
& 0.2y_1 + 0.1y_2 + 4.2y_3 + 28.7y_4 \le 0.50 \\
& y_1, y_2, y_3, y_4 \ge 0
\end{aligned}
$$

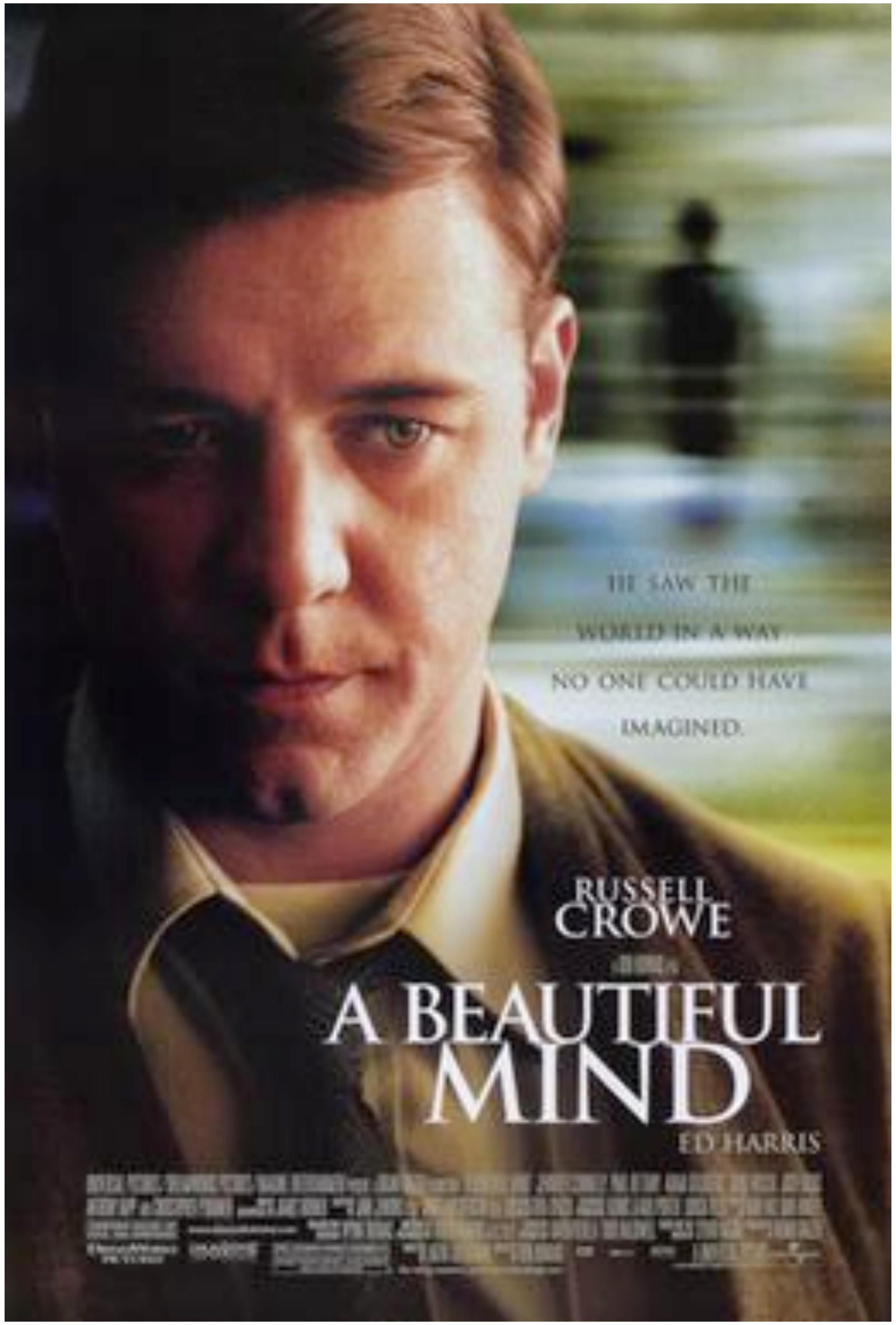
Optimal Solution: £2.27

Duality

- Every linear mathematical **minimisation** problem has an associated **maximisation** problem.
- This problem is called the **dual**.
- A problem and its dual always have the same optimal value.

Game Theory

- Covered in last 2-3 weeks of the module.
- We can imagine this as a sort of “game” between the student and the vitamin profiteer.
- In general, this leads to a theory of **decision making** in **competition** with another agent.
- This is called “Game Theory” and is a major topic in economics (worth 11 Nobel prizes!)\
- Invented by John Nash


HE SAW THE
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RUSSELL
CROWE
A BEAUTIFUL
MIND
ED HARRIS

Example 10.1. Suppose that Rosemary and Colin each have 2 cards, labelled with a 1 and a 2. Each selects a card and then both reveal their selected cards. If the sum s of the numbers on their cards is even, then Rosemary wins and Colin must pay her this s. Otherwise, Colin wins and Rosemary must pay him s.

- We will see how to mathematically model such games and use linear programming to find optimal strategies for both players.
- What is the best strategy for each player? Is this a fair game?

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- We will see how to mathematically model such games and use linear programming to find optimal strategies for both players.

- In this example, we can show that the optimal strategy for Rosemary is to select her cards **randomly:**
    - Card 1 with probability 7/12
    - Card 2 with probability 5/12

- We can show that this game is **unfair** and the best that Rosemary can do will require her to pay Colin 1/12 in expectation.

Module Information

Assessment

- 20% of grade from 3 Courseworks worth 6.67% each (due every 3-4 weeks)
  - 1 or 2 coursework questions set each week
  - all solutions to these will to be submitted every 3-4 weeks (deadlines given on QMplus).
  - Must be submitted on QMplus as PDF (legible scans / photos OK).
  - Late submissions **not accepted**.
  - Coursework graded within 2 weeks with feedback.

- 80% of grade from final exam
  - This will be an **in-person** exam **3 hour** exam.
  - 2 past papers with solutions will be made available on QMplus.
  - Other past exam questions will be covered in seminars.

Teaching Schedule

- **Lectures**
  - Thursday 15:00-18:00 (Arts Two LT + zoom (if it works))
  - Zoom recordings and/or Q-review recordings on QMplus
  - Will try to break up the lectures with
    - breaks
    - small activities / quizzes

- **Seminars** (check your timetable)
  - Will cover "seminar questions" including past exam questions.
  - Try to look at some seminar questions before the seminar
  - Opportunity to get feedback on your work and ask questions

Composition

- **Breakdown of module**
  - New concepts (~20%)
  - Modelling (~15%)
  - algorithms / procedures / methods (~40%)
  - Proofs (~25%)

- Warning: First proof comes up in Week 4
- Small list of non-examinable content given at the end

Resources

- Lecture notes
  - Complete printed lecture notes from past years on QMplus (longer / more detailed explanations)
  - Handwritten notes / slides posted each week on QMplus (shorter / more direct)
- Recordings of lectures available on QMplus.
- Recommended books / articles **(completely optional)**

More Help

- Seminars
- Student forum on QMplus for questions
    - If something in the notes or lectures is unclear to you, then probably others are having the same problem!
    - You can answer each other's questions.
- Can email questions to me at viresh.patel@qmul.ac.uk
- Learning Support Hour: Tuesdays 14:00-15:00 in the School Social Hub (MB-B11)
- Ask questions in lectures / during break

Expectations

- Follow material from each week
  - Attend/watch lectures
  - Read board/printed notes
  - Look at checklist

- Engage with seminars
  - Briefly look at seminar questions in advance
  - Attend seminar, attempt seminar questions
  - **Ask for feedback / ask questions**
  - Make sure you understand solutions

- Coursework
  - Attempt weekly coursework exercises
  - Submit all these exercises by given deadline

QMplus

Rest of the Lecture

- Review of Linear Algebra
- Notation and conventions used throughout the module
- Key definitions for linear programs.