MTH5114 **Linear Programming and Games, Spring 2023**

Week 1 Seminar Questions **Viresh Patel**

Sample Question (From 2019 Exam): Rewrite the following linear program in *standard inequality form*:

$$\begin{aligned} \text{minimize} \quad & -3x_1 + 6x_2 + 9x_3 + 12x_4 \\ \text{subject to} \quad & x_1 + x_2 - x_4 \geq 3, \\ & x_1 + x_2 - 3x_4 \leq 5, \\ & 3x_1 + 2x_2 + x_3 = 9, \\ & x_1, x_2 \geq 0, \\ & x_4 \leq 0, \\ & x_3 \text{ unrestricted} \end{aligned}$$

If $(x_1, x_2, x_3, x_4) = (a, b, c, d)$ is an optimal solution for the linear program above, write down an optimal solution for the linear program in standard form that you have found in the first part of the question. (You may have to consider some cases.)

Questions for discussion Questions 1 and 2 are for revision of linear algebra in preparation for some of the proofs in the module (around week 4). Questions 3 and 4 test how well you have understood the definition of optimal solution.

1. (a) Show that the vector $\mathbf{y} = A\mathbf{x}$ is a linear combination of the columns of A.
   (b) Show that if a square matrix A is invertible then its columns must be linearly independent.

2. Determine whether or not each of the following vectors is linearly independent:
   (a) $\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$
   (b) $\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 8 \\ 8 \end{pmatrix}$
   (c) Any set of m vectors of length n where $m > n$.

3. When defining linear programs, we have assumed that the objective function is a linear combination $\mathbf{c}^\mathsf{T}\mathbf{x}$, and so the solution $\mathbf{x} = \mathbf{0}$ always has objective value 0. What if we wanted to maximise a quantity such as: $3x_1 + 2x_2 - 4x_3 + 10$? More generally, suppose we want to solve a problem of the form:

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^\mathsf{T}\mathbf{x} + k \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{1}$$

where k is a constant.

(a) Find a linear program whose optimal solution $\mathbf{x}$ is the same as the optimal solution for the above problem.

(b) Prove that your program has this property (that is, that its optimal solution is the same as the optimal solution for the given mathematical program).

4. Consider an arbitrary linear program in standard inequality form. Determine which of the following transformations will change the optimal solution to this program:

(a) Multiplying the objective by a non-negative constant k.

(b) Multiplying the matrix A by a non-negative constant k.

(c) Multiplying the matrix $\mathbf{b}$ by a non-negative constant k.

(d) Multiplying both A and $\mathbf{b}$ by a non-negative constant k.