Introduction to Vectors and Matrices

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January2024

Why vectors and matrices?

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots\cdots \times \mathbb{R}}_{n\text{ - times}}$$

How is the module organised

2-hour lecture on Monday (Me)

2-hour lecture on Thursday (Matthew)

(1 + 1)

Tutorials

Online-test : $\underbrace{3, 5, 8, 10, 12}$ $\rightarrow$ 20%

Exam (in person) 80%

How are the lecture notes organised

SMS EDI Committee

Learning Caffé Claudia

Tuesday 1–2

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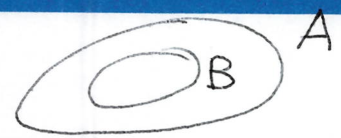
Elements of Logic and Set theory

$x \in A$



Set theoretic symbols

- $\in$ is in, is an element of
- $\subset$ is a subset of, is contained in $B \subset A$
- $\subseteq$ subset (possibly equal) $B \subseteq A$
- $\cup$ union $A \cup B = \{x : x \in A \text{ or } x \in B\}$ $A \subset A \cup B$, $B \subset A \cup B$
- $\cap$ intersection $A \cap B = \{x : x \in A \text{ and } x \in B\}$
- $\emptyset$ the empty set. $\emptyset$ $\circ$

Question: $A \cup B = \emptyset$?

We can also reverse these symbols or negate them.

For instance, the reverse of $\subset$ is $\supset$ which means contains and the negation of $\in$ is obtained by striking out the symbol, i.e., $\notin$. x does not belong to A

$x \notin A$

This last symbol stands for *is not an element of*.

$A \subset B$ $A \supset B$

$\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$

$\mathbb{N} = \{1, 2, 3, \dots\}$

Logical symbols

- $\forall$ for all — $\forall x \in \mathbb{N}, \quad x^2 \in \mathbb{N}$
- $\exists$ there exists — $\exists x \in A$; ......
- $\nexists$ there does not exist
- $\Rightarrow$ implies, is a sufficient condition for
- $\not\Rightarrow$ does not imply
- $\Leftrightarrow$ if and only if, iff — $A \Longrightarrow B$, $B \Longleftarrow A$

For all natural numbers, the square is still a natural number

These symbols are quantifiers ($\forall$, $\exists$, ...)

or express logical relations ($\Rightarrow$, $\Leftrightarrow$, ...).

If a natural number is even then that number plus 1 is odd.

$x = 2k, \quad k \in \mathbb{N}$ even

$x = 2k \Longrightarrow x + 1 = 2k + 1$ odd

Set theory

Every mathematical object belongs to a *set*.

Definition

A set is a collection of objects. The objects are referred to as *elements* of the set.

- We can represent a set by listing its elements or by stating a property that determines membership or in other words defines the set.

Example: the set of numbers 1, 2, 3 is denoted as

$$S = \{1, 2, 3\}.$$

The set of all positive numbers (numbers greater than 0) can be represented by stating its defining property, i.e.,

$$T = \{x : x > 0\},$$
$$T = \{x | x > 0\}.$$

Note that : and | read as *such that*.

More definitions and notations

- If x is an element of S then we write $x \in S$.
- If x does not belong to S then we write $x \notin S$.
- S is a subset of T if every element of S belongs to T, i.e.,

$$x \in S \Rightarrow x \in T.$$

- If S is a subset of T we write $S \subset T$. If the set might actually be equal then we can use the notation $S \subseteq T$. We can write $T \supset S$ or $T \supseteq S$.
- The empty set $\emptyset$ is the set with no elements.
- Let A and B be two sets. The union of A and B is the set of all x such that $x \in A$ **or** $x \in B$. In detail,

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

- The intersection of A and B is $A \cap B = \{x : x \in A \text{ and } x \in B\}$.
- The set of all elements of A which does not belong to B is denoted by $A \setminus B$.

Remark

It is clear from the definition above that $A \cap B$ could be empty. Moreover,

$$\begin{aligned} A \cap B &\subseteq A, \\ A \cap B &\subseteq B, \\ A &\subseteq A \cup B, \\ B &\subseteq A \cup B, \\ A \cap B &\subseteq A \cup B, \\ A \setminus B &\subseteq A. \end{aligned}$$

Exercise

Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 5\}$. Compute $A \cup B$, $A \cap B$ and $A \setminus B$. What is the relation between $(A \cup B) \setminus (A \cap B)$ and $A \setminus B$?

Sets of numbers

$\mathbb{N} = \{1, 2, 3, 4, \cdots\}$ natural numbers

$\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$ natural numbers with 0 included

$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \cdots\}$ integers

$\mathbb{Q} = \{a/b : \ a \in \mathbb{Z}, \ b \in \mathbb{N}\}$ rational numbers

$$\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q}$$

$\mathbb{R}^2$ $\mathbb{R}^3$

$\mathbb{R}^n$

$$\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

Logic

proposition

Definition

A *statatement* is a sentence which is either true or false.

Examples

All the people in this room wear glasses

All natural numbers when multiplied give natural numbers

- A statement can contain a *variable*, such as
  "The number x is positive".
  Here x is a variable, and the statement can be true or false depending on the value of x. True for $x = 1$, false for $x = -2$.

Combining statements

Using statements with variables one can form new statements in several ways:

(i) Using the quantifier $\forall$: from the statement $b^2 \geq \frac{1}{2}$ one can form the statement

$$\boxed{\forall b \in \mathbb{N}, \text{one has } b^2 \geq \frac{1}{2}.}$$

This statement is true.

(ii) Using the quantifiers $\exists$: $\boxed{\exists b \in \mathbb{Q} \text{ such that } b^2 \geq \frac{1}{2}.}$

(iii) Using the implication $\Rightarrow$:

$$\boxed{b^2 \geq \frac{1}{2} \quad \Rightarrow \quad b^2 \geq \frac{1}{3}.}$$

(iv) Using the double implication $\Leftrightarrow$:

$$\boxed{b^2 \geq \frac{1}{2} \quad \Leftrightarrow \quad -b^2 \leq -\frac{1}{2}.}$$

$\forall$ $\exists$

quantifiers

Let $b \in \mathbb{N}$.
Then, $b \geq 1$.
Then, $b^2 \geq 1 \geq \frac{1}{2}$. $\square$

Negating statements

- If a statement is true then its negation is false and vice versa.
- When negating statements containing symbols $\exists$ and $\forall$, one should replace $\exists$ by $\forall$ and vice versa.
- When negating statements containing the symbol $\Rightarrow$ one should replace $\Rightarrow$ by $\not\Rightarrow$.

Example

if $x \geq 2$ then $x+1 \geq 3$

statement A: $\boxed{\forall x \in \mathbb{N}\ x \geq 2 \Rightarrow x+1 \geq 3.}$

$x \geq 2$

$x+1 \geq 2+1$

True

$\neg A$ $\exists x \in \mathbb{N}: \ x \geq 2 \not\Rightarrow x+1 \geq 3$ False

$\neg A$ $\exists x \in \mathbb{N}: \ x \geq 2 \Longrightarrow x+1 < 3$ False

Equivalence and implication

Let A and B be two statements which contain the same variable.

Then $A \Leftrightarrow B$ means that B is true if and only if A is true.

Examples

$\forall m \in \mathbb{Z}$ $\quad \overset{A}{m \text{ is even}} \Longleftrightarrow \overset{B}{m^2 \text{ is even}}$

$A \Longrightarrow B$

$B \Longrightarrow A$

$A \Rightarrow B$ means 'if A is true then B is true'.

Examples

If $\underbrace{k \in \mathbb{N}}_{A}$ then $\underbrace{k \in \mathbb{Z}}_{B}$ $\quad A \Longrightarrow B$

$A \Rightarrow B$

1. A implies B.
2. If A then B.
3. A is a sufficient condition for B.
4. B is a necessary condition for A.

- We say that $A \Rightarrow B$ is a conditional statement.
- Note that it can happen that $A \Rightarrow B$ but NOT $B \Rightarrow A$.

Examples:

Some basic mathematical language

Definition ✓

Theorem ✓

Proposition

Corollary

Lemma

Proof — A collection of logical steps that prove that a theorem holds.

Exercise:

A: $\forall x \in \mathbb{Q} \quad \exists k \in \mathbb{N} \quad x \leq k$. — Hypothesis

B: $\forall x \in \mathbb{Q} \quad \exists k \in \mathbb{N} \quad x+2 \leq k$. — Thesis

Prove that $A \Rightarrow B$

Proof: Let $x \in \mathbb{Q}$. I know by hypothesis that $\exists k \in \mathbb{N} \quad x \leq k$.

It follows that $x+2 \leq k+2$.

This shows that $\exists\, k' = k+2 \in \mathbb{N} : \quad x+2 \leq k'$ $\square$