MTH 4104 Example Sheet IV

Shu SASAKI

IV-1. What are the quotients and remainder when X^5 is divided by $X^2+(1+\sqrt{-1})X+\sqrt{-1}$ in $\mathbb{C}[X]$.

IV-2. Given an example of a ring R and a polynomial f in $R[X]$ such that the number of solutions in R is greater than $\deg(f)$.

IV-3. If x and y are elements of a ring R such that $xy = yx$, we say that x and y commute, or x commutes with y. Find all 2-by-2 matrices with coefficients in $\mathbb{R}$ which commute with $\begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}$.

IV-4. Let R be a ring and n be a positive integer. Describe the rings $\mathrm{M}_n(R[X])$ and $\mathrm{M}_n(R)[X]$. Given an example of an element of each ring with $n = 2$ and $R = \mathbb{Z}$. Explain how these two rings are related to each other.

IV-5. Let R be a ring. Prove (R×1) for $\mathrm{M}_2(R)$.

IV-6. Let R be a ring. Prove that the set $\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in R \right\}$ of 2-by-2 matrices over R whose lower left entry is zero is a ring, with the usual addition and multiplication of matrices. You may assume that $\mathrm{M}_2(R)$ is a ring.

IV-7. Label the corners of a regular pentagon 1 through 5 in order. Each of the symmetries of the pentagon– reflections, rotations, and the identity (doing nothing)– gives a permutation in S_5 describing how it moves the corners. (a) Write down the set of all these permutations. (b) Is your set closed under composition?

IV-8. A permutation matrix in $\mathrm{M}_n(\mathbb{R})$ is a matrix so that, in each row and column, there is exactly one 1, and all other entries are 0. For example, the identity matrix is a permutation matrix. (a) Pick some $n \geqslant 3$ and write down two more permutation matrices in $\mathrm{M}_n(\mathbb{R})$ other than the identity matrix. Then compute their product. (b) Why is the name 'permutation matrix' appropriate?