Relativity

MTH 6132 Course notes

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Contents

Preface

These are notes for the Relativity course (MTH6132) I am currently lecturing during the Spring 2022 term at the School of Mathematical Sciences of Queen Mary, University of London. The material is primarily based on both typeset and handwritten notes I have inherited from Dr. Juan Valiente-Kroon and Dr Shabnam Beheshti; the primary addition I am making to these notes is the discussion on black holes and gravitational waves (not implemented yet).

The present course on Relativity is aimed at mathematics and physics undergraduates interested in learning the mathematical foundations of Special and General Relativity. In particular, very little physical background is assumed, so a certain amount of time is spent presenting underlying assumptions and experimental motivation for such a theory. The lectures also assumes minimal prerequisites on the mathematical side. Necessary ideas from differential geometry and tensors are self-contained and references are provided for further study. The course is quite an ambitious one, divided approximately into thirds. It begins with Special Relativity, then moves to Differential Geometry and finally it provides an introduction to General Relativity.

Due to time constraints, there are some clear omissions in the choice of topics. In particular, in the chapter on Special Relativity it would be desirable to have a discussion of the Maxwell equations. In the chapter on Differential Geometry, it would be elegant to discuss the Hamilton-Jacobi Equations as motivation for generalizing curvature. In the chapter on General Relativity, the discussion is restricted to the vacuum field equations with little mention of the field equations with matter. Also, it would be desirable to include positive mass theorems, cosmological models and of course, the most recent and exciting progress on gravitational waves! Thorough mathematical investigation of these topics would require at the very least a couple of weeks more. Perhaps a re-orgainsation of the topics and thoughtful collaboration could result in a two-semester course in Geometric Analysis and General Relativity. I do not discard the possibility of carrying out such a revision in future iterations of the course. In the meantime, please know that the current lecture notes have been adapted to my particular understanding and appreciation of the subject.

Corrections, omissions and suggestions for improvements with which the readers of these notes may favour me will be greatly appreciated.

–P. Figueras

Chapter 1

Introduction

1.1 What is Relativity?

The term Relativity encompasses two physical theories proposed by Einstein[1]. Namely, *Special Relativity* and *General Relativity*. However, as we will see, the word relativity is also used in reference to *Galilean Relativity*[2]. The term *Theory of Relativity* was first coined by Max Planck[3] in 1906 to emphasize how a theory devised by Einstein in 1905 —what we now call Special Relativity— uses the *Principle of Relativity*.

1.1.1 Special Relativity?

Special Relativity is the physical theory of the measurement in *inertial frames of reference*. It was proposed in 1905 by Albert Einstein in the article *On the Electrodynamic of moving bodies* (*Zur Elektrodynamik bewegter Körper*, Annalen der Physik **17**, 891 (1905)). It generalises Galileo's Principle of Relativity —all motion is relative and that there is no absolute and well-defined state of rest. Special Relativity incorporates the principle that the speed of light is the same for all inertial observers, regardless the state of motion of the source. The theory is termed *special* because it only applies to the special case of inertial reference frames —i.e. frames of reference in uniform relative motion with respect to each other. Special Relativity predicts the equivalence of matter and energy as expressed by the formula

$$E = mc^2.$$

Special Relativity is a fundamental tool to describe the interaction between elementary particles, and was widely accepted by the Physics community by the 1920's.

1.1.2 General Relativity?

General Relativity is the geometric theory of *gravitation* published by Albert Einstein in 1915 in the article *The field equations of Gravitation* (*Die Feldgleichungen der Gravitation*, Sitzungsberichte der Preussischen Akademie der Wisenschaften zu Berlin 884). It generalises Special Relativity and Newton's law of universal gravitation, providing a unified description of gravity as the manifestation of the curvature of spacetime. The theory is *general* because it applies the Principle of Relativity to any frame of reference so as to handle general coordinate transformations. From General Relativity it follows that

[1] Albert Einstein (1879-1955). Physicist of German origin. Died in the USA.
[2] Galileo Galilei (584-1642) . Italian physicist, mathematician and astronomer.
[3] Max Planck (1858-1947). German physicist.

Special Relativity still applies locally. The domain of applicability of General Relativity is in Astrophysics and Cosmology. More recently, the Global Positioning System (GPS) requires of General Relativity to function accurately! Contrary to Special Relativity, General Relativity was not widely accepted until the 1960's.

1.2 Pre-relativistic Physics

1.2.1 Galilean Relativity

In order to study General Relativity one starts discussing Special Relativity. To this end, it is important to briefly look at pre-relativistic Physics to see how Special Relativity arose.

The starting point of Special Relativity is the study of motion. For this one needs the following ingredients:

- **Frames of reference.** These consist of an origin in space, 3 orthogonal axes and a clock.
- **Events.** This notion denotes a single point in space together with a single point in time. Thus, events are characterised by 4 real numbers: an ordered triple (x, y, z) giving the location in space relative to a fixed coordinate system and a real number giving the Newtonian time. One denotes the event by $E = (t, x, y, z)$.

There are an infinite number of frames of reference. Motion relative to each frame looks, in principle, different. Hence, it is natural to ask: is there a subset of these frames which are in some sense simple, preferred or natural? The answer to this question is yes. These are the so-called *inertial frames*. In an inertial frame an isolated, non-rotating, unaccelerated body moves on a straight line and uniformly.

Inertial frames are not unique. There are actually an infinite number of these. This raises the question: can one tell in which inertial frame are we in? It turns out that within the framework of Newtonian Mechanics this is not possible. More precisely, one has the following:

Galilean Principle of Relativity. Laws of mechanics cannot distinguish between inertial frames. This implies that there is no absolute rest. In other words, the laws of Mechanics retain the same form in different inertial frames.

In this sense, Relativity predates Einstein.

1.2.2 Laws of Newton

The three Laws of Newtonian Mechanics[4] are:

(1) Any material body continues in its state of rest or uniform motion (in a straight line) unless it is made to change the state by forces acting on it. This principle is equivalent to the statement of existence of inertial frames.

(2) The rate of change of momentum is equal to the force.

(3) Action and reaction are equal and opposite.

[4] Isaac Newton (1643-1727). English physicist and mathematician.

These laws or principles, together with the following fundamental assumptions (some of which are implicitly assumed in Newton's laws) amount to the *Newtonian framework*:

(A1) Space and time are continuous —i.e. not discrete. This is necessary to make use of the Calculus.

(A2) There is a universal (absolute) time. Different observers in different frames measure the same time. In fact, Newton also regarded space to be absolute as well. However, the absoluteness of space is not necessary for the development of the Newtonian framework, as space intervals turn out to be invariant under Galilean transformations. Historically, Newton demanded this for subjective reasons.

(A3) Mass remains invariant as viewed from different inertial frames.

(A4) The Geometry of space is Euclidean. For example, the sum of angles in any triangle equals 180 degrees.

(A5) There is no limit to the accuracy with which quantities such as time and space can be measured.

As it will be seen in the sequel, Assumptions 2 and 3 are relaxed in Special Relativity while Assumption 4 is relaxed in General Relativity. Assumption 5 is relaxed in Quantum Mechanics —not to be discussed in the course. Presumably Assumption 1 will be relaxed in Quantum Gravity!

1.2.3 Newtonian mechanics

The aim of this section is to provide a *concise* summary of the Newtonian equations describing the motion of point-like particles. The trajectory of each particle is described by a curve determining its position at each instant t. For instance the uniform motion mentioned in Newton's first law above corresponds to the trajectory

$$x(t) = v_x t + b_x \ , \quad y(t) = v_y t + b_y \ , \quad z(t) = v_z t + b_z \ , \tag{1.1}$$

where v_i and b_i are constants. The time derivative of the trajectory yields the velocity of the particle at each time

$$\underline{v} = \frac{d\underline{x}}{dt} \quad \Rightarrow \quad v_x = \frac{dx}{dt} \ , \quad v_y = \frac{dy}{dt} \ , \quad v_z = \frac{dz}{dt} \ . \tag{1.2}$$

Similarly the acceleration is the variation of the velocity

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d^2\underline{x}}{dt^2} \quad \Rightarrow \quad a_x = \frac{d^2x}{dt^2} \ , \quad a_y = \frac{d^2y}{dt^2} \ , \quad a_z = \frac{d^2z}{dt^2} \ . \tag{1.3}$$

It is easy to check that in the case (1.1) the velocity is constant and the acceleration vanishes, as expected. As another example, let us consider a circular motion in the plane (x, y), which is more easily described by introducing polar coordinates $x = \rho\cos\phi$, $y = \rho\sin\phi$

$$\rho(t) = \rho_0 \ , \quad \phi(t) = \Omega t + \phi_0 \ , \tag{1.4}$$

where ρ_0 is the (constant) radius of the circular motion, Ω is the (constant) angular frequency and ϕ_0 is another constant determining the position of the particle at $t = 0$. In this case the acceleration is non-vanishing. By using (1.4) it is straightforward to check that $\underline{a} = -\Omega^2 \underline{x}$.

By Newton's second law stated above, the acceleration in an inertial frame is due to a physical force $\underline{F}$ acting on the particle

$$\underline{F} = m\underline{a} = \frac{d}{dt}\underbrace{(m\underline{v})}_{\underline{p}} , \tag{1.5}$$

where in the second step we used (1.3) and the Newtonian assumption that m is constant. The quantity $\underline{p}$ in the round parenthesis is called (linear) momentum.

It is interesting to show that Newton's third law stated above is equivalent to the statement that the total momentum of an isolated system is conserved. Let us see how this works for the case of two point-like particles. By indicating with $\underline{F}_1$ the force that the second particle exerts on the first and with $\underline{F}_2$ the force that the first particle exerts on the second, we have that $\underline{F}_1 + \underline{F}_2 = 0$ by Newton's third law. Then, by using (1.5), we have

$$\underline{F}_1 + \underline{F}_2 = 0 \quad \Rightarrow \quad \frac{d}{dt}(\underline{p}_1 + \underline{p}_2) = 0; , \tag{1.6}$$

proving that the sum of the momentum of the two particle does not change in time, i.e. it is conserved.

In this course we are mainly interested in the gravitational force, so let us state Newton's formula for this force between two point-like objects separated by a distance $|\underline{x}_1 - \underline{x}_2|$:

$$\begin{aligned} \underline{F}_1 &= -G\frac{m_1 m_2}{|\underline{x}_1 - \underline{x}_2|^2}\left(\frac{\underline{x}_1 - \underline{x}_2}{|\underline{x}_1 - \underline{x}_2|}\right) , \\ \underline{F}_2 &= G\frac{m_1 m_2}{|\underline{x}_1 - \underline{x}_2|^2}\left(\frac{\underline{x}_1 - \underline{x}_2}{|\underline{x}_1 - \underline{x}_2|}\right) , \end{aligned} \tag{1.7}$$

where $\underline{F}_i$ is the force acting on the particle $i = 1, 2$. Two comments: $\sum_{i=1}^{2} \underline{F}_i = 0$ in agreement with Newton's third law and notice that the force is always attractive, since the round parenthesis in both expressions is a unit vector point from particle 2 to particle 1 (so $\underline{F}_1$ is oriented towards particle 2 because of the overall minus, while $\underline{F}_2$ is oriented towards particle 1). A further comment: the masses appearing in (1.7) are the same as those appearing in (1.5)! This need not to be the case and, in principle, we could have introduced two different concepts: the inertial mass in (1.5) and the gravitational mass in (1.7). The fact that they are the same implies that the acceleration of a particle due to the gravitational is independent of its mass. Finally, Newton's theory for the gravitational force is *linear*, which means that the gravitational force acting on a particle is simply the sum of the gravitational forces due to all other particles present in the system under study. We will see that this feature is *not* shred by Einstein's theory of gravitation, making the latter much more challenging.

It is possible to derive the gravitational force (1.7) from a scalar function known as the potential energy

$$U = -G\frac{m_1 m_2}{|\underline{x}_1 - \underline{x}_2|} \quad \Rightarrow \quad \underline{F}_1 = -\nabla_{\underline{x}_1} U , \tag{1.8}$$

where ∇ is the gradient. Explicitly it reads

$$(F_1)_x = -\frac{\partial U}{\partial x_1} , \quad (F_1)_y = -\frac{\partial U}{\partial y_1} , \quad (F_1)_z = -\frac{\partial U}{\partial z_1} \tag{1.9}$$

and similarly for the force $\underline{F}_2$ acting on the particle 2 where the derivatives are with respect to the position (x_2, y_2, z_2). Since the gravitational force can be deduced by a potential, it is *conservative*. This means that it is possible to define a quantity, the total energy, which does not change in time. By combining (1.8) and (1.5) we have

$$\begin{aligned}
&\underline{v} \cdot \frac{d}{dt}(m\underline{v}) = \underline{v} \cdot \underline{F} = -\underline{v} \cdot \nabla_{\underline{x}} U(\underline{x}) \;, \qquad \Rightarrow \\
&\frac{d}{dt}\left(\frac{m}{2}\underline{v}^2\right) = -\frac{d}{dt} U(\underline{x}) \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{m}{2}\underline{v}^2 + U(\underline{x})\right) = 0 \;,
\end{aligned} \tag{1.10}$$

which implies that the sum of the kinetic and the potential energy $\frac{m}{2}\underline{v}^2 + U(\underline{x})$ is conserved.

For future applications it is useful to introduce the (Newtonian) gravitational field $\underline{g}$. Consider a test particle of mass m placed in the position $\underline{x}$, then $\underline{g}(\underline{x})$ is defined as

$$\underline{g} \equiv \frac{\underline{F}}{m} \;. \tag{1.11}$$

Thanks to the discussion above, it is immediate to see that $\underline{g}$ can be derived from a scalar function $\phi(\underline{x})$ called the (Newtonian) gravitational potential and

$$\underline{g}(\underline{x}) = -\nabla_{\underline{x}} \phi(\underline{x}) \;. \tag{1.12}$$

From (1.8) it is easy to see that the gravitational potential generated by a point-like particle of mass M placed at the origin of the Cartesian system is

$$\phi_M(\underline{x}) = -G\frac{M}{|\underline{x}|} \;. \tag{1.13}$$

You can check that $\phi_M(\underline{x})$ satisfies

$$\nabla^2 \phi_M(\underline{x}) \equiv \frac{\partial^2 \phi_M(\underline{x})}{\partial x^2} + \frac{\partial^2 \phi_M(\underline{x})}{\partial y^2} + \frac{\partial^2 \phi_M(\underline{x})}{\partial z^2} = 0 \;, \quad \text{if} \quad \underline{x} \neq 0 \;. \tag{1.14}$$

Notice that the equation above becomes singular at point $\underline{x} \neq 0$. We can make sense of this point as well by using the Dirac delta function

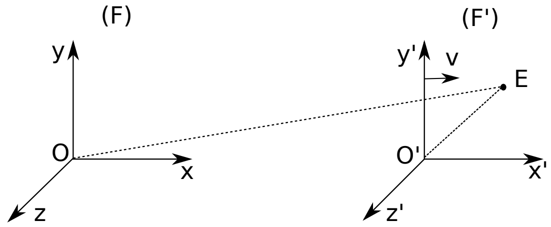
$$\nabla^2 \phi_M(\underline{x}) \equiv \frac{\partial^2 \phi_M(\underline{x})}{\partial x^2} + \frac{\partial^2 \phi_M(\underline{x})}{\partial y^2} + \frac{\partial^2 \phi_M(\underline{x})}{\partial z^2} = 4\pi G \rho(\underline{x}) \;, \quad \text{with} \quad \rho(\underline{x}) = M\delta^3(\underline{x}) \;. \tag{1.15}$$

Here $\rho(\underline{x})$ is the mass distribution which is fully concentrated in $\rho(\underline{x}) = 0$ for the point-like case, *i.e.* the total mass $M \int d^3\underline{x}\, \rho(\underline{x})$ takes contributions only from $\underline{x} = 0$. Let us stress that the first equation in (1.15) is valid for a generic (continuous) mass distribution (essentially thanks to the linearity of Newton's theory for the gravitational force mentioned above).

1.2.4 Galilean transformations

Galilean transformations tell us how to transform from one inertial frame to another.

Consider two inertial frames: $F(x, y, z, t)$ and $F'(x', y', z', t')$ moving with velocity $\underline{v}$ relative to one another in *standard configuration* —that is, F' moves along the x axis of the frame F with uniform speed v; all axes remain parallel. See the figure:



Now, suppose that at a given moment of time t, an event E is specified by coordinates (t, x, y, z) and (t', x', y', z') relative to the frames F and F', respectively. Let the origins O and O' coincide at $t = 0$. From the figure one sees that

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t. \tag{1.16}$$

In the more general case of inertial frames of reference where the velocity has also components in the y and z axes one has:

$$\underline{r}' = \underline{r} - \underline{v}t,$$

where $\underline{v} = (v_x, v_y, v_z)$ and $\underline{r} = (x, y, z)$, $\underline{r}' = (x', y', z')$ are the position vectors with respect to the frames F and F', respectively. Notice that in the case of frames of reference in standard configuration one has $v_y = v_z = 0$).

Remark. It is customary to call the observer associated to the inertial frame F, *Joe*, and that of F', *Moe*.

1.2.5 Galilean transformation formulae for the velocity and acceleration

To see this, let the position of a particle P be specified by $\underline{r} = \underline{r}(t)$ relative to a frame F. The motion relative to F' is given by equation (1.16). Differentiating both sides twice with respect to t (notice that $t = t'$) gives:

$$\underline{V}' = \underline{V} - \underline{v}, \tag{1.17a}$$
$$\underline{a}' = \underline{a}, \tag{1.17b}$$

where

$$\underline{V} = \frac{\mathrm{d}\underline{r}}{\mathrm{d}t}, \quad \underline{a} = \frac{\mathrm{d}^2\underline{r}}{\mathrm{d}t^2},$$

are, respectively, the velocity and acceleration of the particle with respect to the frame F while

$$\underline{V}' = \frac{\mathrm{d}\underline{r}'}{\mathrm{d}t}, \quad \underline{a}' = \frac{\mathrm{d}^2\underline{r}'}{\mathrm{d}t^2},$$

are the velocity and acceleration of the particle with respect to F'.

Remark. Notice that as a consequence of the transformation formula for the acceleration (1.17b), the acceleration of the particle as measured by F and F' coincide. Thus, although the position and the velocity are different in each system of reference, both sets of observers agree on the acceleration. This result is some times phrased as: *acceleration is Universal.*

Example. The following example will be of relevance in the sequel. Consider a cannonball moving along the x-axis. If the cannonball has velocity V with respect to the frame

F, then the velocity as measured by the frame F' (moving with velocity v with respect to F) is given by $V' = V - v$. In what follows, suppose for simplicity that $v > 0$. Then if $V > 0$ (i.e. the cannonball moves away from the origin of F) then $V' = V - v < V$ —that is, F' sees the cannonball moving more slowly. On the other hand if $V < 0$ (the cannonball goes towards the origin of F), then $|V - v| > v$ so that F' sees the cannonball moving faster.

1.2.6 Invariance of Newton's laws under Galilean transformations

Important for the sequel is the notion of invariance. Invariance refers to properties of a system that remain unchanged under a particular type of transformations. In the previous section we have already seen that two inertial systems of reference measure the same acceleration of a moving particle. Thus, acceleration is an invariant under Galilean transformations.

In what follows, we will see that the laws of Mechanics keep the same form as we go from one inertial frame to another —i.e. under Galilean transformations. The First and Third Laws are invariant as the former involves inertial frames and the latter involves accelerations which are invariant. It remains to show that the Second Law (the fundamental equation of Newtonian Mechanics)

$$m\frac{\mathrm{d}\underline{V}}{\mathrm{d}t} = m\underline{a} = \underline{f} \tag{1.18}$$

is invariant as we go from one inertial frame to another.

To show the invariance of (1.18) recall that $\underline{a}' = \underline{a}$ and m remains invariant (by assumption) so that one only needs to show that $\underline{f}$ remains invariant as we go from F to F'. To do this, recall that generally $\underline{f}$ takes the form $\underline{f} = \underline{f}(\underline{r}, \underline{v}, t)$ where usually $\underline{r}$ and $\underline{v}$ are the relative distance and the relative velocity between two bodies. One can verify that the relative distances and velocities remain invariant. That is,

$$\underline{V}_2' - \underline{V}_1' = \underline{V}_2 - \underline{V}_1, \quad \underline{r}_2' - \underline{r}_1' = \underline{r}_2 - \underline{r}_1.$$

This implies that $\underline{f}$, and hence the Second Law remains invariant under changes in the inertial frames.

This discussion amounts to a form of self-consistency, in the sense that Physics, when confined to Newtonian Mechanics, satisfies the Galilean Principle of Relativity.

1.2.7 Electromagnetism

Special Relativity arises from the tension between Newtonian Mechanics with the other great physical theory of the 19th century —Electromagnetism. The fundamental laws of Electromagnetism are the so-called *Maxwell equations*[5] :

$$\begin{aligned}
&\nabla \cdot \underline{D} = \rho, \\
&\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}, \\
&\nabla \cdot \underline{B} = 0, \\
&\nabla \times \underline{H} = \underline{j} - \frac{\partial \underline{D}}{\partial t},
\end{aligned}$$

[5] James C. Maxwell (1831-1879). Scottish mathematician.

where $\underline{B}$ is the magnetic induction, $\underline{E}$ the electric field, $\underline{H}$ the magnetic field, $\underline{D}$ the electric displacement, $\underline{j}$ the electric current and ρ the electric charge. In vacuum, the relations between them are $\underline{D} = \varepsilon_0 \, \underline{E}$ and $\underline{B} = \mu_0 \, \underline{H}$, where ε_0 is the vacuum permitivity and μ_0 is the vacuum permeability.

It can be shown that these equations predict the existence of electromagnetic waves for $\underline{E}$ and $\underline{H}$ in the form

$$\nabla^2 \underline{E} = \frac{1}{c^2} \frac{\partial^2 \underline{E}}{\partial t^2}, \quad \nabla^2 \underline{H} = \frac{1}{c^2} \frac{\partial^2 \underline{H}}{\partial t^2},$$

where $c = 1/\sqrt{\varepsilon_0 \mu_0}$ is the speed of propagation of the waves. These electromagnetic waves were soon identified with the propagation of light.

We recall that light travels with a speed of $c \approx 3 \times 10^8 m/s$ [6]. This was first measured by Rømer [7] in 1675 by studying the delay in the appearance of moons of Jupiter. It is of interest to noticed that fastest object created by Mankind, a satellite probing the Sun, had a speed of $70km/s$ which corresponds to about 0.0002% of the speed of light!

Within the Newtonian framework, the Maxwell equations give rise to two problems:

(1) With respect to which system of reference is the speed of light c is measured? First, it was assumed that the absolute space of Newton —the so-called *ether*— was the medium in (and relative to) which light moved. However, attempts at detecting the effects of Earth's motion on the velocity of light —the so-called terrestrial ether drift— all failed. The most important of these was the *Michelson-Morley experiment*[8]. This gave a null result.

(2) It is easy to show that Maxwell's equations and the wave equation do not remain invariant under Galilean transformations.

These problems gave to a crisis in the 19th century Physics. Three scenarios were put forward to resolve the tension. These were:

(i) Maxwell's equation were incorrect. The correct laws of Electromagnetism would remain invariant under Galilean transformations.

(ii) Electromagnetism had a preferred frame of reference —that of ether.

(iii) There is a Relativity Principle for the whole of Physics —Mechanics and Electromagnetism. In that case the laws of Mechanics need modification.

Now, Electromagnetism was very successful and have a very strong predictive power. There was no experimental support for (ii). Hence the point of view (iii) was adopted by Einstein. His resolution of the tension between Mechanics and Electromagnetism came to be known as Special Relativity.

[6] The letter c used to denote the speed of light comes from the Latin word *celeritas*, velocity, speed.
[7] Ole C. Rømer (1644-1710). Danish astronomer.
[8] Albert Michelson (1852-1931). Edward Morley (1838-1923). American physicists.

Appendix: General Galilean transformations

In general, if the coordinate axes are not in standard configuration and the origins O and O' of the coordinate axes do not coincide, then the general form of the transformation takes the form:

$$\underline{r}' = R\underline{r} - \underline{v}t + \underline{d},$$

where R is the rotation matrix aligning the axes of the frames and $\underline{d}$ is the distance between the origins at $t = 0$. Note that the general transformation is linear, so that F' is inertial if F is. The most general transformation would also include

$$t' = t + \tau$$

where τ is a real constant.

These transformations form a 10-parameter group (1 for τ, 3 for $\underline{v}$, 3 for $\underline{d}$, and 3 for R). The group property implies that the composition of two Galilean transformations is a Galilean transformation, and that given a Galilean transformation there is always an inverse transformation. The Galilean transformations restricted to standard configurations form a 1-parameter subgroup of this group, with v as variable.

Chapter 2

Special Relativity

The contradiction brought about by the development of Electromagnetism gave rise to a crisis in the 19th century that Special Relativity resolved.

2.1 Einstein's postulates of Special Relativity

(i) There is no ether (there is no absolute system of reference).

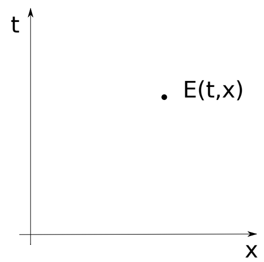
(ii) The laws of Nature have the same form in all inertial frames (Einstein's principle of Relativity)

(iii) The velocity of light in empty space is a universal constant, i.e. same for all observers and light sources, independent of their motion —Michelson & Morley's result is promoted to an axiom.

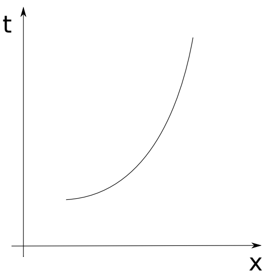
Note that postulate (iii) is clearly incompatible with Galilean transformations which imply $c' = c - v$. Because of this the Galilean transformations need modification. This leads to the *Lorentz transformations.*

2.2 Spacetime diagrams

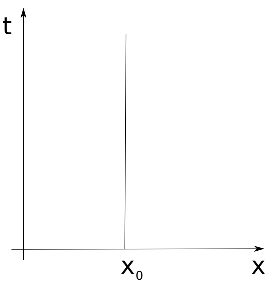
Spacetime is defined as the set of 4 reals (t, x, y, z). An **event** in spacetime is represented by a point $E(t, x, y, z)$. For simplicity (in order to be to visualise) confine ourselves to 2 dimensions: one space and one time coordinates so that events are depicted by $E(t, x)$. Such diagrams are a very useful way to approach problems involving multiple frames of reference.



The **wordline** of a particle is defined as the set of all points that the trajectory of a particle follows in spacetime.



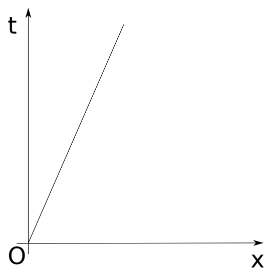
To develop our intuition, we consider a few examples. The worldline of a particle which is stationary at $x = x_0$ is a vertical line:



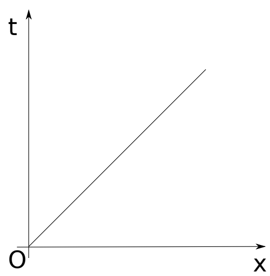
The worldline of a particle moving with uniform velocity v and passing through O at $t = 0$ is straight line:

$$x = vt \quad \text{so that} \quad t = \frac{1}{v}x.$$

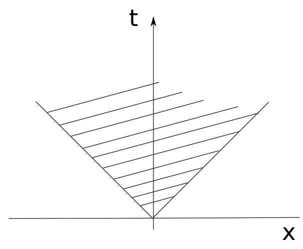
Therefore the slope of of the line is given by $1/v$.



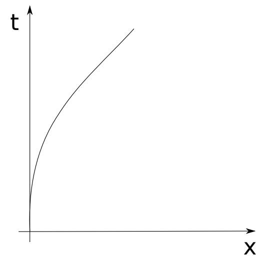
The worldline of a light ray is a straight line with slope equal to $1/c$. In practice we shall usually choose $c = 1$ so that the slope is equal to 1.



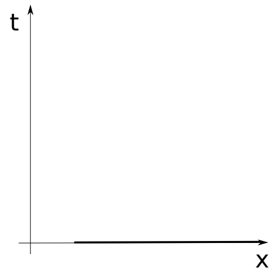
Note. All uniformly moving particles have worldlines which are straight lines with slopes *bigger* than $1/c$ or bigger than 1 if $c = 1$. Therefore they all lie in the shaded region of the figure.



The worldlines of accelerating bodies are curved. For example, for a uniformly accelerated body from rest one has that initially the worldline is tangent to the t. The upper bound for v is c. The slope of the asymptotic motion is $1(= 1/c)$. This situation will be analysed in detail later on.

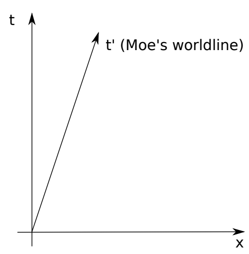


The worldlines of instantaneous travel is a horizontal line —however, this is forbidden within the framework of Special Relativity.



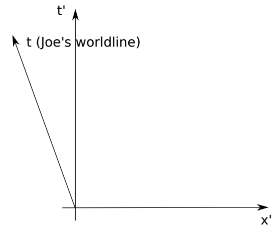
Some further examples

The following example is based on the notion that every particle in uniform motion (with velocity less than the speed of light) is an inertial frame of reference. Let F denote the frame of reference associated to Joe. Then if Moe moves with velocity v with respect to Joe, one has the following diagram:



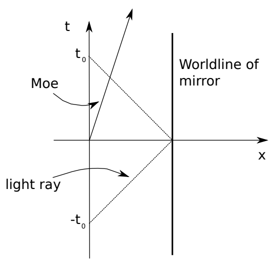
An important observation is that the t'-axis coincides with Moe's worldline.

Now, from the point of view of Moe, Joe moves away with velocity $-v$. One has the following diagram:



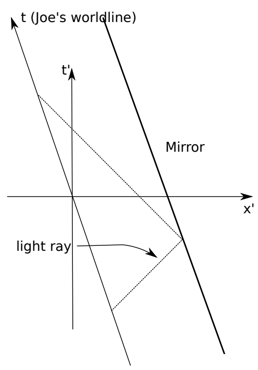
Notice that the t-axis coincides with Joe's worldline.

As a final example, we consider the following situation: a light ray is shot at from $x = 0$ at $t = -t_0$ in the positive direction of the x-axis. The light ray reflects at a mirror located at $x = x_0$ and returns to $x = 0$ at $t = 0$. The corresponding diagram is:



Notice that from Joe's (system F), the time the light ray requires to go back and forth are equal.

Now, we consider the diagram of the situation as seen by the system of reference F' (Moe). The required diagram is given by



Notice that the light rays in this diagram are also lines with slope of 45 degrees as required by the postulates of relativity. Notice that from Moe's point of view the times requires by the light ray to go back and forth are not equal!

2.3 Lorentz transformations (LT)

In this section we address the following question: what type of transformation does one need to ensure that the speed of light as measured by two inertial frames of reference F and F' is equal?

In order to explore the consequences of this requirement, let us consider 2 inertial systems of reference in standard configuration moving with relative velocity v. Suppose a light ray is fired at $x = 0$ at $t = 0$. Futhermore, suppose that this light ray reaches (t, x). Let (t', x') be the coordinates of the event (t, x) as seen by F'. As the speed of light is c for both systems of reference, one has that

$$c = x/t, \quad c = x'/t'.$$

For reasons that will become clear later in the chapter, it is covenient to rewrite these expressions as

$$0 = -c^2t^2 + x^2, \quad 0 = -c^2t'^2 + x'^2.$$

Thus one has that

$$-c^2t^2 + x^2 = -c^2t'^2 + x'^2. \tag{2.1}$$

One can readily verify (by direct substitution) that the Galilean transformation $t = t'$, $x = x' - vt$, cannot satisfy this condition. Thus, one needs to consider a different kind of transformation.

The so-called *Lorentz transformations* are given by:

$$x' = \gamma(x - vt), \quad t' = \gamma\left(t - \frac{vx}{c^2}\right), \quad y' = y, \quad z' = z, \quad \text{with} \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{2.2}$$

Remark. This is a particular case of a more general transformation with 10 parameters. These parameters are the 3 components of the velocity, 3 components of a shift of the origin, 3 parameters of a rotation and a further parameter fixing the origin of the time. The set of these transformations forms a group. The transformation given by (2.2) is the 1-parameter subgroup of this group called the *special Lorentz group*.

Remark. One can verify by direct substitution that the Lorentz transformation (2.2) satisfies

$$-c^2t^2 + x^2 = -c^2t'^2 + x'^2.$$

Remark. It is interesting what happens with the Lorentz transformations for low velocities. Using a Taylor expansion about 0, we recall

$$(1 - x)^{-1/2} = 1 + \tfrac{1}{2}x + O(x^2).$$

It follows that

$$\gamma \approx 1 + \tfrac{1}{2}\frac{v^2}{c^2}.$$

Now, if $v \ll c$, then $v^2/c^2 \approx 0$, so that $\gamma \approx 1$. Hence, form the expersions for the Lorentz transformation one has that

$$t' \approx t, \qquad x' \approx x - vt.$$

That is, one recovers the Galilean transformations!

The inverse Lorentz transformation

We have discussed the Lorentz transformation which given the coordinates (t, x) of an event as seen by the system of reference F, allows to compute the coordinates (t', x') as seen by F'. Now, we are interested in the inverse transformation which given (t', x') allows to calculate (t, x). By symmetry, as F and F' are both inertial systems, the inverse transformation should have the same functional form. The key observation is then that if F sees F' moving with velocity v, then F' sees F moving with velocity $-v$. Hence, the required transformtion is given by

$$
\begin{aligned}
t &= \gamma\left(t' + \frac{vx'}{c^2}\right), \\
x &= \gamma(x' + vt').
\end{aligned}
$$

Remark. One could also have obtained the required expressions by inverting directly the original Lorentz transformation formulae. This is, however, a much longer computation! A similar short argument an be used for the transformation formulae for the velocity and the acceleration.

2.4 Clocks and rods in relativistic motion

We now consider the effects of uniform motion on clocks and rods.

2.4.1 Time dilation

Consider F and F' in standard configuration. Let a standard clock be at rest in F' (at $x = x_0$) and consider two events in this clock at times t'_1 and t'_2. Let also

$$\Delta t' = t'_2 - t'_1.$$

In order to find the interval Δt as measured by F, recall that

$$\Delta t = \gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right).$$

However, $\Delta x' = 0$ as $x'_2 = x'_1 = x_0$. Hence one obtains

$$\Delta t = \gamma \Delta t',$$

Since

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} > 1,$$

one finds that the interval as measured by F is longer.

There is a symmetry! Both observers say the same thing about each other!

2.4.2 Length contraction

This is also called the (Lorentz-Fitzgerald contraction). Consider F and F' in standard configuration. Let a rod of length $\Delta x'$ be placed at rest along the x'-axis of F'. To find the length as measured in F, we must measure the distance between the two ends of the rod simultaneously in F. Consider two events occurring simultaneously at the end points of the rod in F. Therefore one has $\Delta t = 0$. Now, using

$$\Delta x' = \gamma(\Delta x - v\Delta t)$$

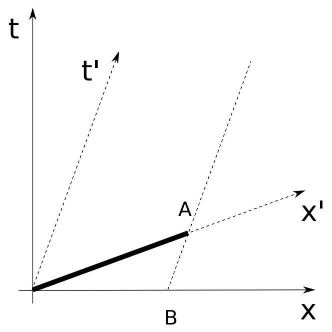
one finds that

$$\Delta x' = \gamma \Delta x, \qquad \text{or} \qquad \Delta x = \frac{1}{\gamma}\Delta x'.$$

Accordingly, the length of the rod in the direction of motion as measured by F is reduced by a factor of $(1 - v^2/c^2)^{1/2}$.

Geometrically:

F measures the distance between the two ends of the rod at $t = 0$, i.e. F measures OB, while F' measures OA.



2.5 Paradoxes

These arise from an incautious view of the situation, and the fact that simultaneity means different things to different observers.

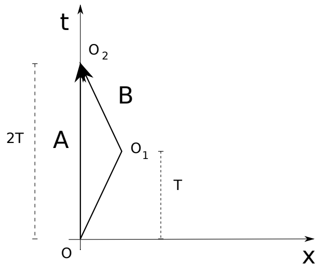
The twin paradox

Consider a pair of twins A and B. Let A be stationary at origin of F whereas B moves with sped v for a time T and then with speed $-v$ for equal time and returns to A's position. The total elapsed time as measured by A is $2T$. Because of time dilation, the time as measured by B is

$$\frac{2T}{\gamma} < 2T.$$

Therefore, when twins reach the point $(0, 2T)$ in A's frame A is older than B.

The "paradox": cannot B say with equal right that it was she/he who remained where she/he was while A went on a round trip and that A should, consequently, be the younger when they meet?



Answer: No, since there is no symmetry! The twin A remained in the same inertial frame, but B has experienced acceleration and deceleration and therefore knows that she/he has not been in an inertial frame! This solves the paradox.

2.6 Experimental evidence for Special Relativity

Clearly Special Relativity is consistent with Michelson & Morley's experiment and its refined versions since.

A well know test of time dilation comes from the behaviour of muons (elementary particles formed by the collision of Cosmic rays with particles in the upper atmosphere). The mean life of muons is approximately $2.2 \times 10^{-6}s$ so that if the moved at the speed of light they could only cover a distance of approximately $0.66km$. However, they reach the ground level from heights of about $10km$. To explain this, they must have a dilation factor of approximately 15. This means they would have a speed of about $0.997c$!

From the muon's point of view, they have a normal life time, however, they depth of the atmosphere is contracted by a factor of 15,

Time dilation can also be observed using accurate atomic clocks on board of airplanes which are then compared with fixed clocks.

2.7 Hyperbolic form of the Lorentz transformations

This a convenient representation for showing the group properties of the Lorentz transformation.

The key idea is to replace the velocity parameter v by a hyperbolic parameter α that satisfies the following:

$$\cosh\alpha = \gamma, \quad \sinh\alpha = \frac{v}{c}\gamma, \quad \tanh\alpha = \frac{v}{c}.$$

We also require α and v to have the same sign as $\cosh\alpha = \cosh(-\alpha)$.

The Lorentz transformation (2.2) becomes (*hyperbolic form of the Lorentz transformation*):

$$x' = x\cosh\alpha - ct\sinh\alpha, \tag{2.3a}$$
$$ct' = -x\sinh\alpha + ct\cosh\alpha, \tag{2.3b}$$
$$y' = y, \tag{2.3c}$$
$$z' = z \tag{2.3d}$$
$$\tag{2.3e}$$

Adding and subtracting x' and ct' as given by (2.3a) and (2.3b) one obtains

$$ct' + x' = e^{-\alpha}(ct + x), \tag{2.4a}$$
$$ct' - x' = e^{\alpha}(ct - x), \tag{2.4b}$$

where it has been used that

$$\cosh\alpha = \frac{e^{\alpha} + e^{-\alpha}}{2}, \quad \sinh\alpha = \frac{e^{\alpha} - e^{-\alpha}}{2}.$$

To show that the Lorentz transformations form a group one needs to show:

(i) there exists an identity element;

(ii) for every Lorentz transformation there exists an inverse;

(iii) the composition of Lorentz transformations is a Lorentz transformation and that the composition is associative.

The most convenient way to verify the latter is to use the form given by (2.4a) and (2.4b) and then check one by one:

(i) One sees that there exists an identity Lorentz transformation corresponding to v ($\alpha = 0$).

(ii) There exists an inverse Lorentz transformation with $v = -v$ ($\alpha \to -\alpha$).

(iii) Let F'' move with velocity v_2 (α_2) relative to F' and F' with velocity v_1 (α_1) relative to F —all in standard configuration.

From (2.4a) and (2.4b) one has that

$$\begin{aligned} &ct'' + x'' = e^{-\alpha_2}(ct' + x'), \\ &ct'' - x'' = e^{\alpha_2}(ct' - x'), \\ &y'' = y, \quad z'' = z' \end{aligned}$$

and

$$\begin{aligned} &ct' + x' = e^{-\alpha_1}(ct + x), \\ &ct' - x' = e^{\alpha_1}(ct - x), \\ &y' = y, \quad z' = z. \end{aligned}$$

It follows then that

$$\begin{aligned} &ct'' + x'' = e^{-(\alpha_1+\alpha_2)}(ct + x), \\ &ct'' - x'' = e^{(\alpha_1+\alpha_2)}(ct - x), \\ &y'' = y, \quad z'' = z', \end{aligned}$$

which shows that the composition of Lorentz transformations is a Lorentz transformation and since the hyperbolic parameters add, one also has the associativity.

The previous discussion allows also to discuss the Special Relativity rule for the *composition of velocities*. Since the resultant of two Lorentz transformations with parameters α_1 and α_2 is a Lorentz transformation with parameters $\alpha_1+\alpha_2$, the corresponding relation between the velocity parameter of the transformation can be easily derived from

$$\tanh\alpha = \frac{v}{c}$$

by recalling that

$$\tanh(\alpha_1+\alpha_2) = \frac{\tanh\alpha_1 + \tanh\alpha_2}{1+\tanh\alpha_1\tanh\alpha_2}.$$

Substituting for

$$\tanh\alpha_1 = \frac{v_1}{c}, \quad \tanh\alpha_2 = \frac{v_2}{c}, \quad \tanh(\alpha_1+\alpha_2) = \frac{v}{c}$$

one obtains

$$v = \frac{v_1+v_2}{1+v_1v_2/c^2} \tag{2.5}$$

where v is the velocity of F'' relative to F —it represents the *relativistic sum of collinear velocities v_1 and v_2 along the x-axis.* A generalisation of this rule will be discussed later.

Remark 1. When

$$\frac{v_1}{c} \ll 1, \quad \frac{v_2}{c} \ll 1,$$

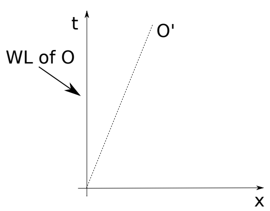
then equation (2.5) takes the Galilean form

$$v = v_1 + v_2.$$

Remark 2. Since $|\tanh\alpha| < 1$, it follows that v always satisfies $|v| < c$.

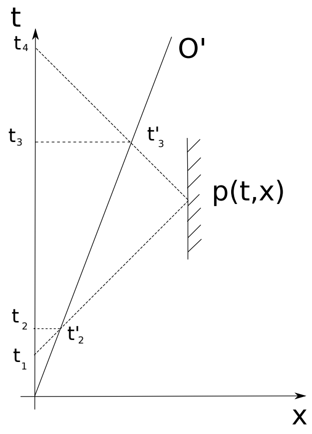
Appendix: Derivation of the Lorentz transformations

Consider two frames F and F' moving in standard configuration —i.e. O' moves with speed v along the x-axis relative to O. The worldline of O' in the frame is given as in the figure:



Let observers O and O' carry clocks measuring t and t' respectively such that when O' is at (t, vt) according to O, the clock at O' registers $t' = \beta t$, where β may be a function of v —in this sense β carries all the effect that the motion has on t. Note also that $\beta = 1$ for Galilean transformations.

Now consider a light ray emitted by O at $t = t_1$, travelling via O', being reflected at $p(t, x)$ and received by O at $t = t_4$ —i.e. a round trip.



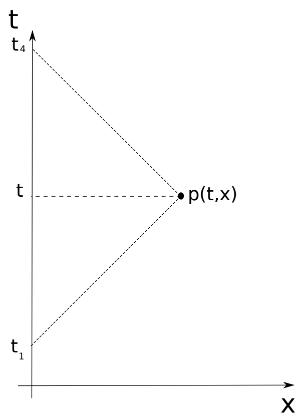
We want to relate the coordinates of the event at p relative to the frames F and F'.

In line with Einstein's postulates assume that the speed of light is c for both O and O'. From the perspective of O the distance and time may be fixed using the so-called *radar convention*:

$$x = \tfrac{1}{2}c(t_4 - t_1), \quad t = \tfrac{1}{2}(t_4 + t_1)$$

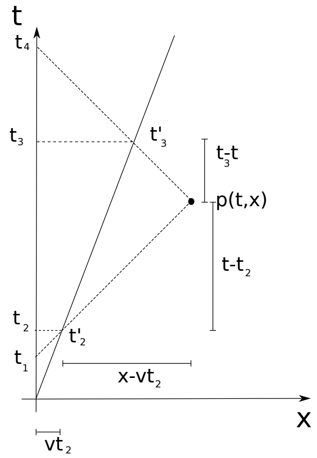
so that

$$x = c(t - t_1) = c(t_4 - t). \tag{2.6}$$



Similarly,

$$x - vt_2 = c(t - t_2), \tag{2.7}$$
$$x - vt_3 = c(t_3 - t). \tag{2.8}$$



Now, equations (2.7) and (2.8) imply, respectively

$$t_2 = \frac{ct - x}{c - v}, \quad t_3 = \frac{ct + x}{c + v}.$$

The corresponding times as measured by O' are:

$$t'_2 = \beta t_2 = \beta\left(\frac{ct - x}{c - v}\right), \tag{2.9a}$$

$$t'_3 = \beta t_3 = \beta\left(\frac{ct + x}{c + v}\right), \tag{2.9b}$$

where it has been used that $t' = \beta t$. Therefore, the time and location of $p(t, x)$ as measured by O' is (using again the radar convention) is given by:

$$x' = \tfrac{1}{2}c(t'_3 - t'_2) = \frac{\beta c^2(x - vt)}{c^2 - v^2}, \tag{2.10a}$$

$$t' = \tfrac{1}{2}(t'_3 + t'_2) = \frac{\beta(c^2 t - vx)}{c^2 - v^2}, \tag{2.10b}$$

where equations (2.9a) and (2.9b) have been used to obtain the second equalities in the last pair of equations.

Note. The observer O' is also assuming that the velocity of light is c. This assumption is inconsistent with the Galilean transformations.

Eliminating x between (2.10a) and (2.10b) one obtains

$$t = \frac{1}{\beta}\left(t' + \frac{vx'}{c^2}\right). \tag{2.11}$$

Now, the *Relativity principle* requires that we obtain the same result if we interchange x, x' and t, t' and let $v \to -v$. Applying this idea to equation (2.10b) and equating to (2.11):

$$t = \frac{\beta(c^2 t' + vx')}{c^2 - v^2} = \frac{1}{\beta}\left(t' + \frac{vx'}{c^2}\right), \tag{2.12}$$

so that

$$\beta = \left(1 - \frac{v^2}{c^2}\right)^{1/2}.$$

Letting $\gamma \equiv 1/\beta$, the transformation for x' can be found from (2.10a):

$$x' = \gamma(x - vt). \tag{2.13}$$

Similarly for t from equation (2.10b):

$$t' = \gamma\left(t - \frac{vx}{c^2}\right).$$

Finally, the coordinates y and z remain the same as there is no motion in these directions.

2.8 Further discussion on the Lorentz Transformations

2.8.1 Transformation formula for the velocity

Let F and F' be in standard configuration and moving with velocity v along the x-axis. For simplicity, we will restrict our attention to movements along the x-axis. Let V be the velocity of a particle relative to F. To find V', the velocity relative to F' recall that:

$$V \equiv \frac{\mathrm{d}x}{\mathrm{d}t}, \tag{2.14a}$$

$$V' \equiv \frac{\mathrm{d}x'}{\mathrm{d}t'}, \tag{2.14b}$$

where the increment represents the distances and times between two events for the particle relative to the two frames. Using the differential form of the Lorentz transformations

$$\mathrm{d}x' = \gamma(\mathrm{d}x - v\mathrm{d}t), \qquad \mathrm{d}t' = \gamma(\mathrm{d}t - v/c^2\mathrm{d}x),$$

in (2.14b) one obtains

$$V' = \frac{dx'}{dt'} = \frac{\gamma(\mathrm{d}x - v\mathrm{d}t)}{\gamma(\mathrm{d}t - v/c^2\mathrm{d}x)} = \frac{V - v}{1 - Vv/c^2}.$$

Remark. If $v \ll c$ one finds that $V' \approx V - v$. That is, one recovers the Galilean transformation. Note also that if we do not restrict our attention to motion in the x-direction, then one computes the other components of the velocity V' in a similar manner to above using $\frac{dy'}{dt'}$ and $\frac{dz'}{dt'}$. Observe that in contrast with the Galilean transformations, the velocity components transverse to the direction of motion of frame F' *are affected by the Lorentz transformation*!

2.8.2 Transformation formula for the acceleration

A similar transformation for the acceleration can be found. Recall that

$$a \equiv \frac{\mathrm{d}V}{\mathrm{d}t}, \qquad a' \equiv \frac{\mathrm{d}V'}{\mathrm{d}t'}.$$

Starting from

$$V' = \frac{V - v}{1 - Vv/c^2}$$

and calculating the differential

$$\mathrm{d}V' = \frac{\mathrm{d}V}{1 - Vv/c^2} + \frac{V - v}{(1 - Vv/c^2)^2} v/c^2\mathrm{d}V,$$

one concludes that

$$\mathrm{d}V' = \frac{1}{\gamma^2}\frac{\mathrm{d}V}{(1 - vV/c^2)^2}. \tag{2.15}$$

Also, from the Lorentz transformation

$$\mathrm{d}t' = \gamma(\mathrm{d}t - v/c^2\mathrm{d}x),$$

it follows that

$$\frac{\mathrm{d}V'}{\mathrm{d}t'} = \frac{1}{\gamma^3(1 - vV/c^2)^3}\frac{\mathrm{d}V}{\mathrm{d}t}.$$

Alternatively, on can write

$$a' = \frac{1}{\gamma^3(1 - vV/c^2)^3}a.$$

Notice that as a consequence of this formula, although acceleration is not an invariant, if the acceleration is zero in one inertial frame, then it is zero in all inertial frames. Hence, acceleration is in a certain sense absolute.

Remark. As before, if $v \ll c$, then one finds that $a' \approx a$ —the Galilean invariance of acceleration.

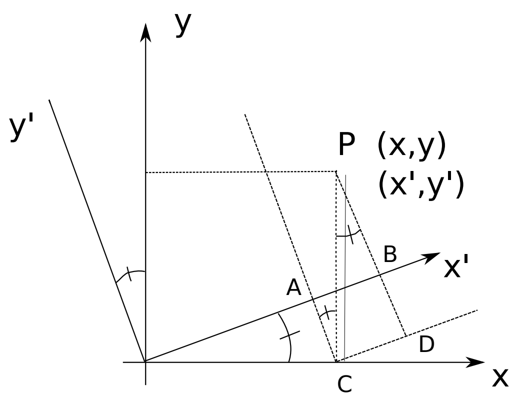
2.9 The Minkowski spacetime

There are many ways to study Special relativity. Here we take the geometrical approach developed in 1908 by H. Minkoswki. This approach naturally leads to (and led Einstein!) to General Relativity.

To gain some intuition, start with the Euclidean geometry of the 2 dimensional plane and recall the transformation of coordinates corresponding to the rotation of Cartesian axes by an angle α in such a plane:

$$\begin{aligned} x' &= x\cos\alpha + y\sin\alpha, \\ y' &= -x\sin\alpha + y\cos\alpha, \end{aligned}$$

where (x, y) and (x', y') correspond to the coordinates of the point p in the two frames.



The transformation can be deduced from the diagram by observing that:

$$\begin{aligned} x' &= OA + AB = OA + CD \\ &= OC\cos\alpha + PC\sin\alpha \\ &= x\cos\alpha + y\sin\alpha \\ y' &= PB = PD - BD \\ &= PC\cos\alpha - OC\sin\alpha \\ &= -x\sin\alpha + y\cos\alpha. \end{aligned}$$

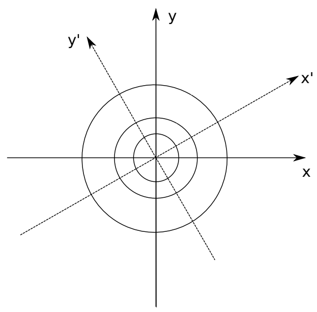
Eliminating the rotation parameter α by taking

$$\begin{aligned} x'^2 + y'^2 &= (x\cos\alpha + y\sin\alpha)^2 + (-x\sin\alpha + y\cos\alpha)^2 \\ &= x^2 + y^2. \end{aligned}$$

Letting

$$(OP)^2 \equiv x^2 + y^2, \tag{2.16}$$

one sees that in Euclidean space, rotations leaves the distance (OP) *invariant.* Note also that the rotation leaves curves of constant distance from the origin —i.e. circles— invariant.



Analogue for Lorentz transformations. Starting from

$$\begin{aligned} ct' + x' &= e^{-\alpha}(ct + x), \\ ct' - x' &= e^{\alpha}(ct - x), \end{aligned}$$

and multiplying both sides one obtains

$$-ct^2 + x^2 = -ct'^2 + x'^2,$$

where the choice of sign in the previous equation is a convention. Furthermore, since $y' = y$ and $z' = z$ one obtains

$$-c^2t^2 + x^2 + y^2 + z^2 = -c^2t'^2 + x'^2 + y'^2 + z'^2. \tag{2.17}$$

Alternatively, one could start from the infinitesimal version of the Lorentz transformations

$$\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right), \quad \Delta x' = \gamma\left(\Delta x - v\Delta t\right), \quad \Delta y' = \Delta y, \quad \Delta z' = \Delta z,$$

and taking the limit in equation (2.17) one obtains

$$-c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = -c^2\mathrm{d}t'^2 + \mathrm{d}x'^2 + \mathrm{d}y'^2 + \mathrm{d}z'^2. \tag{2.18}$$

Therefore

$$-c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

remains invariant under Lorentz transformations (boosts).

Remark 1. The value of c is unit dependent. Often, relativists choose units (relativistic units) such that $c = 1$. That is, distance is measured in *light seconds* —the distance travelled by light in 1 second. *From now on we shall put $c = 1$.* Subsequent formulae may be put "right" dimensionally by putting the missing c's back on basis of dimensional grounds.

Remark 2. With $c = 1$ one has that equation (2.18) reduces to

$$-\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

which, apart from the negative sign is very similar to the Euclidean distance in 4 dimensions

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{d}w^2.$$

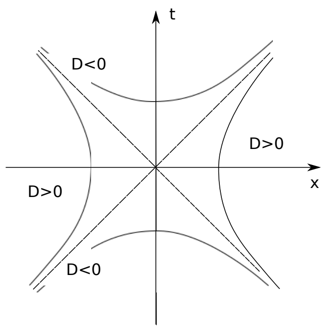
Furthermore, they both remain invariant under coordinate transformations: Lorentz transformations and rotations, respectively. This invariant quantity is called the *interval* $\mathrm{d}s^2$ (or *line element*) in a new type of geometry called the *Minkowski geometry* or *spacetime*. It is then described by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

The latter measures the "distance" between events (t, x, y, z) and $(t + \mathrm{d}t, x + \mathrm{d}x, y + \mathrm{d}y, z + \mathrm{d}z)$ in spacetime.

Note. As opposed to Euclidean geometry, the set of points with equal distances from the origin defines a *hyperbola*:

$$x^2 - t^2 = D, \quad D \text{ a constant.}$$



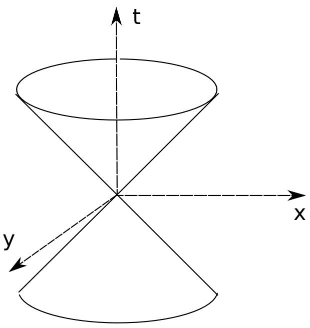
The set of curves in Minkowski space that are left invariant by Lorentz transformations are hyperbolae.

The light cone

In the sequel the 3-dimensional surface in 4-dimensional spacetime given by

$$x^2 + y^2 + z^2 - c^2t^2 = 0, \tag{2.19}$$

will be of importance. This is said to define a *light cone* at the origin, because all lights rays emitted at $t = 0$ at origin lie on the cone $x^2 + y^2 + z^2 = c^2t^2$. Suppressing 1-space dimension one has the following figure:



This figure is obtained by noticing that for constant t, equation (2.19) describes a sphere or radius $|t|$. Notice that the radius of the sphere increases as $|t|$ increases. Also, notice that if $y = z = 0$ one obtaines lines in the (t, x) plane with slope of ± 45 degrees.

2.9.1 Minkowski diagrams

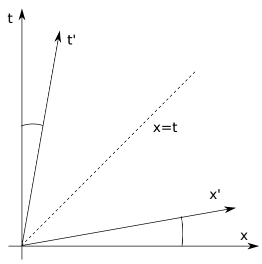
The consequence of Special Relativity are best visualised using Minkowski diagrams. These are pictures in Minkowski spacetime (usually $x - t$ pictures). As an example let us look at the positions of the x' and t' axes relative to the x and t axes.

The x' axis (i.e. $t' = 0$) is given by $(c = 1)$:

$$t' = \gamma(t - vx), \quad \text{so that} \quad t = vx.$$

Similarly, the t' axis (i.e. $x' = 0$) is given by

$$x' = \gamma(x - vt) = 0, \quad \text{so that} \quad t = \frac{1}{v}x$$



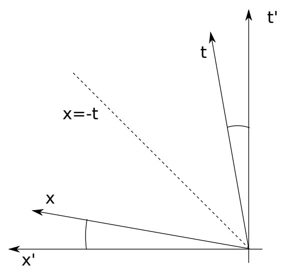
One can also ask what is seen in the reference frame F'. For this one can use the inverse Lorentz transformations

$$t = \gamma(t' + vx'), \quad x = \gamma(x' + vt').$$

The x and t axes from the point of view of the frame F' are given, respectively, by

$$t' = -vx', \quad t' = -\frac{1}{v}x'.$$

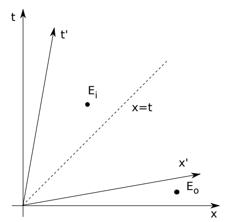
Thus, the picture from F''s point of view is the following:



This picture is consistent with the Principle of Relativity —all frames of reference are equivalent and should provide an equivalent picture! We shall see further examples of this symmetry in the sequel.

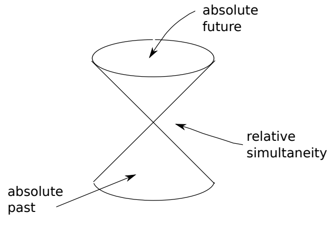
2.9.2 A brief discussion of causality

In what follows we discuss some consequences of the x-dependence of the Lorentz transformation of time.



Consequence 1. Any event E_i inside the light cone occurring after O from the perspective of F will also occur after O from the perspective of F' no matter how fast F' moves with respect to F so long as $v \leq c$. An event E_0 outside the light cone and occurring after O from the point of view of F could occur before O from the point of view of F'. Therefore, outside the future (and similarly the past) light cone of O there exists no ordered time sense of events.

Given any point O, the spacetime is divided up into *the absolute past of* O (the past light cone at O) and the *absolute future of* O (the future light cone at O) and a region (spacelike) know as the *region of relative simultaneity.*



Consequence 2. For invariance of causality, interactions must take place at speeds less than c. To see this, consider a process in which an event E_1 causes an event E_2 at super-light speed $u > c$ relative to some frame F. Choose coordinates in F such that E_1 and E_2 occur on the x-axis and let their space and time separation $\Delta x > 0$, $\Delta t > 0$ (i.e. E_1 precedes E_2). Now, in frame F' moving with with velocity v relative to F we have:

$$\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) = \gamma\Delta t\left(1 - \frac{uv}{c^2}\right)$$

where

$$u = \frac{\Delta x}{\Delta t}$$

is the speed of propagation. Now, for

$$\frac{c^2}{u} < v < c$$

we would have $\Delta t' < 0$ so that in F' the event E_2 precedes E_1 —i.e. cause and effect are reversed or we have information from receiver to transmitter!

2.10 4-vectors and tensors in Special Relativity

In order to write Newton's laws in the Minkoswki spacetime, we require 4-vectors. In analogy with 3-vectors (which are invariant under the change of coordinates) we define 4-vectors in the Minkowski 4-dimensional geometry in such a way that the resulting calculus will have equations invariant under Lorentz transformations (boosts). In order to accomplish this, we require index notation, discussed in the next section.

2.10.1 Index notation

In what follows let

$$(t, x, y, z) = (x^0, x^1, x^2, x^3),$$

where the index position is a convention —more about this later. Write

$$x^a, \quad (a = 0,\ 1,\ 2,\ 3)$$

for x^0, x^1, x^2, x^3 we may write (2.18) as

$$\mathrm{d}s^2 = \sum_{a=0}^{3}\sum_{b=0}^{3} \eta_{ab}\mathrm{d}x^a\mathrm{d}x^b \tag{2.20}$$

where η_{ab} is called the *Minkowski metric tensor* given by

$$(\eta_{ab}) \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

so that

$$\eta_{11} = \eta_{22} = \eta_{33} = 1, \quad \eta_{00} = -1,$$

while all other η_{ab}'s are zero.

In order to drop clumsy summations *hereafter* we will use the so-called *Einstein summation convention*:

(i) Whenever an index is repeated (appears exactly twice, once upstairs and once downstairs) in a term, it is understood to imply summation over that index over all its permissible values. We refer to the repeated indices as being "contracted". In this course lower case Latin indices $a,\ b,\ldots$ take values $0,\ 1,\ 2,\ 3$. Hence equation (2.20) may be written

$$\mathrm{d}s^2 = \eta_{ab}\mathrm{d}x^a\mathrm{d}x^b.$$

(ii) Repeated indices as called *dummy indices* since they may be replaced by another index (from the same alphabet!) not already used. For example:

$$\mathrm{d}s^2 = \eta_{ab}\mathrm{d}x^a\mathrm{d}x^b = \eta_{cd}\mathrm{d}x^c\mathrm{d}x^d.$$

(iii) To avoid ambiguity, no index should appear more than twice in the same expression. So, for example,

$$x_i y_i z_i \quad \text{or} \quad x^a y_a z_a$$

are not allowed!

(iv) Indices that occur only once in an expression (or terms of an equation) are called *free indices*. In an equation such indices match in every term. For example consider

$$A^i B_i C_j = D_j.$$

Notice that i is a dummy index and that j is a free index.

Examples

For simplicity in the following examples let the Latin lower case index take values 1, 2.

(1)

$$A^i B^j = \{A^1B^1,\ A^1B^2,\ A^2B^1,\ A^2B^2\}$$

as $i,\ j$ are free indices.

(2)

$$A^i B_i = \sum_{i=1}^{2} A^i B_i = A^1B_1 + A^2B_2,$$

as i is a dummy index.

(3)

$$g_{ij} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

as, again, $i,\ j$ are free indices.

(4) In $\Gamma^i{}_{jk}$ all indices are free. There are 8 terms: $\Gamma^1{}_{11},\ \Gamma^1{}_{12}; \ldots$.

(5) In $R^i{}_{jkl}$ all indices are free and there are 16 terms: $R^1{}_{111},\ R^1{}_{112},\ R^1{}_{122}, \ldots$

(6)

$$\Gamma^i{}_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}s}\frac{\mathrm{d}x^k}{\mathrm{d}s} = \Gamma^i{}_{lm}\frac{\mathrm{d}x^l}{\mathrm{d}s}\frac{\mathrm{d}x^m}{\mathrm{d}s}$$

as $l,\ m$ are dummy indices while i is free.

(7) $x_a y_b z^b = z_a y_c y^c$.

(8) $g_{ij}\mathrm{d}x^i\mathrm{d}x^j = g_{mn}\mathrm{d}x^m\mathrm{d}x^n = g_{11}(\mathrm{d}x^1)^2 + g_{12}\mathrm{d}x^1\mathrm{d}x^2 + g_{21}\mathrm{d}x^2\mathrm{d}x^1 + g_{22}(\mathrm{d}x^2)^2$.

2.10.2 4-vectors

In spacetime, vectors are four-dimensional and that is why we refer to them as 4-vectors. The most important point to remember about vectors is that they are located at a given point in spacetime. You may be used to thinking of vectors are arrows connecting two points on the plane; moreover, one tends to carelessly slide vectors from one point to another. These concepts however do not generalise to curved spaces, where there are no preferred curves connecting two points, or no unique way to move a vector around. Rather, to each point p in spacetime, we associate the set of all possible vectors located at that point; this set is known as the tangent space at p and we denote it by T_p. It is important to think of these vectors as located at a single point p rather than stretching from one point to another.

The tangent space T_p is an abstract vector space for each point in spacetime. Recall that a (real) vector space is a collection of vectors that can be added together and multiplied by real numbers in a linear way. For instance, for any two vectors V and W and real numbers a and b, we have

$$(a+b)(W+V) = a\,V + b\,V + a\,W + b\,W\,.$$

Every vector space has a zero vector that functions as the identity element under vector addition. In some vector spaces there is an additional operation, namely the inner (dot) product, but this requires extra structure.

A vector field is defined as set of vectors with precisely one element at each point in spacetime. It is useful sometimes to decompose vectors into components with respect to some set of basis vectors. A basis is any set of vectors which both spans the vector space and is linearly independent. For any vector space, there are infinitely many bases, but each basis has the same number of vectors; this number is the dimension of the vector space. In Minkowski space, the dimension is of course four.

Consider a basis of four vectors $\hat{e}_{(i)}$ with $i \in [0,1,2,3]$. In a basis adapted to the coordinates x^a, we would have

$$\hat{e}_{(0)} = (1,0,0,0)\,, \quad \hat{e}_{(1)} = (0,1,0,0)\,, \quad \hat{e}_{(2)} = (0,0,1,0)\,, \quad \hat{e}_{(3)} = (0,0,0,1)\,.$$

Any abstract vector A can be written as a linear combination of the basis vectors:

$$A = A^i \hat{e}_{(i)}\,.$$

The coefficients A^i are the components of the vector A. One has to remember that a vector is an abstract geometrical object, while the components are just the coefficients of the vector in some convenient basis.

An example of a vector in spacetime is the tangent vector to a curve. A parametrised curve in spacetime is specified by the spacetime coordinates x^a as a function of the parameter along the curve λ, i.e., $x^a(\lambda)$. Then, the tangent vector $V(\lambda)$ to the curve has components

$$V^a = \frac{dx^a}{d\lambda}\,.$$

The entire vector is given by $V = V^a \hat{e}_{(a)}$. Under a Lorentz transformation the spacetime coordinates x^a change according to

$$x^{a'} = L^{a'}{}_{b}\, x^b \tag{2.21}$$

where $L^{a'}_{\ b}$ is the Lorentz transformation matrix defined as

$$(L^{a'}_{\ b}) \equiv \begin{pmatrix} L^{0'}{}_0 & L^{0'}{}_1 & L^{0'}{}_2 & L^{0'}{}_3 \\ L^{1'}{}_0 & L^{1'}{}_1 & L^{1'}{}_2 & L^{1'}{}_3 \\ L^{2'}{}_0 & L^{2'}{}_1 & L^{2'}{}_2 & L^{2'}{}_3 \\ L^{3'}{}_0 & L^{3'}{}_1 & L^{3'}{}_2 & L^{3'}{}_3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

for the case of two frames in standard configuration. Since the parameter λ along the curve is unchanged by the transformation, we then deduce that the components of the tangent vector must transform as

$$V^a \to V^{a'} = L^{a'}_{\ b} V^b .$$

However, the vector V itself (as opposed to its components in some coordinate system) is invariant under Lorentz transformations. From this, we can deduce the transformation rule for the basis vectors $\hat{e}_{(a)}$ under Lorentz transformations:

$$V = V^a \, \hat{e}_{(a)} = V^{b'} \, \hat{e}_{(b')} = L^{b'}{}_a \, V^a \, \hat{e}_{(b')} \quad \Longrightarrow \quad \hat{e}_{(a)} = L^{b'}{}_a \, \hat{e}_{(b')} .$$

To get the new basis $\hat{e}_{(b')}$ in terms of the old one $\hat{e}_{(a)}$, we should multiply by the inverse of the Lorentz transformation $L^{b'}{}_a$. But the inverse of a Lorentz transformation is another Lorentz transformation, and therefore we can write

$$L^a{}_{b'} \, L^{b'}{}_c = \delta^a_c \, , \quad L^{p'}{}_q \, L^q{}_{r'} = \delta^{p'}_{r'} .$$

Then, the transformation rule for the basis vector is

$$\hat{e}_{(b')} = L^a{}_{b'} \, \hat{e}_{(a)} .$$

Therefore, the set of basis vectors transforms via the inverse Lorentz transformation of the coordinates or vector components.

2.10.3 Dual vectors (one-forms)

Once we have defined a vector space T_p, we can define an associated vector space, namely the dual vector space T_p^* called the co-tangent space. The dual space is the space of all linear maps from the original vector space T_p to the real numbers $\mathbb{R}$. If $\omega \in T_p^*$, V, $W \in T_p$ and a, $b \in \mathbb{R}$, then

$$\omega(a\, V + b\, W) = a\, \omega(V) + b\, \omega(W) \in \mathbb{R} .$$

These maps form a vector space themselves, so if ω and η are dual vectors, then

$$(a\, \omega + b\, \eta)(V) = a\, \omega(V) + b\, \eta(V) .$$

We can similarly introduce a set of basis dual vectors $\hat{\theta}^{(b)}$ defined by

$$\hat{\theta}^{(b)}(\hat{e}_{(a)}) = \delta^b_a .$$

Then every dual vector can be written in terms of its components, which we label with lower indices:

$$\omega = \omega_a \, \hat{\theta}^{(a)} .$$

We will usually write ω_a, in perfect analogy with vectors, to stand for the entire dual vector. Sometimes the elements of T_p are called contravariant vectors and the elements of T_p^* are referred to as covariant vectors or one-forms.

The component notation leads to a very simple way of writing the action of a dual vector on a vector:

$$\begin{aligned}\omega(V) &= \omega_a\, \hat{\theta}^{(a)}(V^b \hat{e}_{(b)}) \\ &= \omega_a\, V^b\, \hat{\theta}^{(a)}(\hat{e}_{(b)}) \\ &= \omega_a\, V^b\, \delta^a_b \\ &= \omega_a\, V^a \in \mathbb{R}\,. \end{aligned} \tag{2.22}$$

Note that we only need to know the components. This equation also suggests that we can think of vectors as linear maps on dual vectors by defining,

$$V(\omega) \equiv \omega(V) = \omega_a\, V^a\,.$$

Therefore, the dual space to the dual space is the original vector space itself.

In spacetime we will be interested in fields of vectors and dual vectors. In that case, the action of a dual vector field on a vector field is not a single number but a scalar (function) on spacetime. A scalar is a quantity that is invariant under Lorentz transformations; it is a coordinate-indepedent map from spacetime to the real numbers.

We can use the same arguments that we used earlier for vectors to deduce the transformation properties of the dual vectors under Lorentz transformations. We find (**exercise**),

$$\omega_{a'} = L^b{}_{a'}\, \omega_b\,, \qquad \hat{\theta}^{(c')} = L^{c'}{}_d\, \hat{\theta}^{(d)}\,.$$

This transformation rules ensure that the scalar $\omega(V)$ is indeed invariant under Lorentz transformations, as it should be.

The simplest example of a dual vector in spacetime is the gradient of a scalar function ϕ,

$$d\phi = \frac{\partial \phi}{\partial x^a}\, \hat{\theta}^{(a)}\,.$$

The conventional chain rule used to transform partial derivatives amounts in this case to the transformation rule of the components of dual vectors:

$$\begin{aligned}\frac{\partial \phi}{\partial x^{a'}} &= \frac{\partial x^b}{\partial x^{a'}} \frac{\partial \phi}{\partial x^b} \\ &= L^b{}_{a'}\, \frac{\partial \phi}{\partial x^b}\,,\end{aligned}$$

where we have used (2.21) for the Lorentz transformation of the coordinates. The fact that the gradient of a scalar function is a dual vector leads to the following shorthand notation for the partial derivatives:

$$\frac{\partial \phi}{\partial x^a} = \partial_a \phi = \phi_{,a}\,.$$

In this lectures we will usually use ∂_a rather than the comma. Note that the gradient acts in a natural way on the example of the vector tangent to a curve:

$$\partial_a \phi\, \frac{dx^a}{d\lambda} = \frac{d\phi}{d\lambda}\,.$$

The result is just the derivative of the function ϕ along the curve $x^a(\lambda)$.

2.10.4 Tensors

Just as a dual vector is a linear map from vectors to $\mathbb{R}$, a tensor T of type (or rank) (k, l) is a multilinear map from a collection of dual vectors and vectors to $\mathbb{R}$:

$$T \,:\, T_p^* \underbrace{\times \cdots \times}_{k\,\text{times}} T_p^* \times T_p \underbrace{\times \cdots \times}_{l\,\text{times}} T_p \to \mathbb{R}\,.$$

Here " $\times$ " denotes the Cartesian product, so for example $T_p \times T_p$ is the space of ordered pairs of vectors. Multilinearity means that the tensor acts linearly in each of its arguments; for instance, for a tensor of type $(1, 1)$, we have

$$T(a\,\omega + b\,\eta, c\,V + d\,W) = a\,c\,T(\omega, V) + a\,d\,T(\omega, W) + b\,c\,T(\eta, V) + b\,d\,T(\eta, W)\,,$$

where ω, $\eta \in T_p^*$, V, $W \in T_p$ and $a, b, c, d \in \mathbb{R}$.

The space of all tensors of fixed type (k, l) forms a vector space. To construct a basis for this space, we first need to define the tensor product, denoted by $\otimes$. If T is a (k, l) tensor and S is an (m, n) tensor, we define a $(k + m, l + n)$ tensor $T \otimes S$ by

$$\begin{aligned} & T \otimes S(\omega^{(1)}, \ldots, \omega^{(k)}, \ldots, \omega^{(k+m)}, V^{(1)}, \ldots, V^{(l)}, \ldots, V^{(l+n)}) \\ = \; & T(\omega^{(1)}, \ldots, \omega^{(k)}, V^{(1)}, \ldots, V^{(l)}) \times S(\omega^{(k+1)}, \ldots, \omega^{(k+m)}, V^{(l+1)}, \ldots, V^{(l+n)})\,. \end{aligned}$$

Here is how it works: first act T on the appropriate set of dual vectors and vectors, and then act S on the reminder and multiply the answers. Note that in general $T \otimes S \neq S \otimes T$.

It is now straightforward to construct a basis for the space of all (k, l) tensors by taking tensor products of basis vectors and dual vectors; this basis will consists of all tensors of the form

$$\hat{e}_{(a_1)} \otimes \cdots \otimes \hat{e}_{(a_k)} \otimes \hat{\theta}^{(b_1)} \otimes \cdots \otimes \hat{\theta}^{(b_l)}\,.$$

In components, we the write an arbitrary (k, l) tensor as

$$T = T^{a_1 \ldots a_k}{}_{b_1 \ldots b_l}\, \hat{e}_{(a_1)} \otimes \cdots \otimes \hat{e}_{(a_k)} \otimes \hat{\theta}^{(b_1)} \otimes \cdots \otimes \hat{\theta}^{(b_l)}\,.$$

Alternatively, one can define the components by acting T on the basis of vectors and dual vectors:

$$T^{a_1 \ldots a_k}{}_{b_1 \ldots b_l} = T(\hat{\theta}^{(a_1)} \otimes \cdots \otimes \hat{\theta}^{(a_k)}, \hat{e}_{(b_1)} \otimes \cdots \otimes \hat{e}_{(b_l)})\,.$$

As with vectors, we will usually denote a tensor T by its components $T^{a_1 \ldots a_k}{}_{b_1 \ldots b_l}$. The action of a general tensor T on a set of vectors and dual vectors is the expected one:

$$T(\omega^{(1)}, \ldots, \omega^{(k)}, V^{(1)}, \ldots, V^{(l)}) = T^{a_1 \ldots a_k}{}_{b_1 \ldots b_l}\, \omega^{(1)}_{a_1} \ldots \omega^{(k)}_{a_k}\, V^{(1) b_1} \ldots V^{(l) b_l}\,.$$

A (k, l) tensor thus has k upper indices and l lower indices. The order is important since it need not act in the same way on its various arguments.

The transformation of the tensor components under Lorentz transformations follows from the transformation properties of the basis of vectors and dual vectors, and it is what one would expect from the placement of the indices:

$$T^{a'_1 \ldots a'_k}{}_{b'_1 \ldots b'_l} = L^{a'_1}{}_{a_1} \ldots L^{a'_k}{}_{a_k} L^{b_1}{}_{b'_1} \ldots L^{b_l}{}_{b'_l} T^{a_1 \ldots a_k}{}_{b_1 \ldots b_l}\,.$$

Although we have defined tensors as linear maps from sets of vectors and dual vectors to $\mathbb{R}$, nothing forces us to act on all the arguments. Thus, a $(1, 1)$ tensor also acts as a map from vectors to vectors:

$$T^a{}_b \,:\, V^b \to T^a{}_b\, V^b\,.$$

Exercise: Check that $T^a{}_b V^b$ transforms as a vector under Lorentz transformations.

Similarly, one can act one tensor on (all or part of) another tensor to obtain a third tensor. For example,

$$U^a{}_b = T^{ad}{}_c S^c{}_{db}\,,$$

is a $(1,1)$ tensor.

In spacetime we have already seen examples of tensors. The Minkowski metric η_{ab} is a $(0,2)$ tensor. The metric provides extra structure to the space and, in particular, it allows us to define an inner product and hence a norm. We have the following operations:

Norm or magnitude of a 4-vector. It is defined by

$$|A|^2 \equiv \eta_{ab} A^a A^b = -(A^0)^2 + (A^1)^2 + (A^2)^2 + (A^3)^2\,, \tag{2.23}$$

which in analogy with the invariance of

$$-t^2 + x^2 + y^2 + z^2 = |\bar{x}|^2, \quad x^a = (t, x, y, z),$$

is invariant.

Example: Show by direct substitution that the norm of a 4-vector is invariant. For simplicity let $c = 1$. One has that

$$\begin{aligned} A'^0 &= \gamma(A^0 - vA^1), \\ A'^1 &= \gamma(A^1 - vA^0), \\ A'^2 &= A^2, \\ A'^3 &= A^3. \end{aligned}$$

Hence,

$$\begin{aligned} -(A'^0)^2 + (A'^1)^2 &= \gamma^2 (A^1)^2 + \gamma^2 v^2 (A^0)^2 - 2\gamma v A^1 A^0 - \gamma^2 (A^0)^2 - \gamma^2 v^2 (A^1)^2 + 2\gamma v A^0 A^1, \\ &= \gamma^2 (A^1)^2 (1 - v^2) - \gamma^2 (A^0)^2 (1 - v^2), \\ &= -(A^0)^2 + (A^1)^2. \end{aligned}$$

Remark. Because of the negative sign in (2.23), the norm of a vector does not have to be positive! A 4-vector A^a is said to be:

- **timelike** if $|A|^2 < 0$,
- **spacelike** if $|A|^2 > 0$,
- **null** if $|A|^2 = 0$.

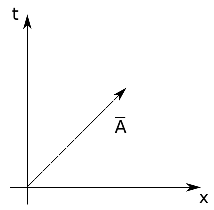
In Minkowski spacetime a null vector need not be a zero vector whose components are zero! Only in a space in which the norm is positive definite, it is true that $|A|^2 = 0$ implies $A^a = 0$.

Example: Show that $A^a = (1, 1, 0, 0)$ is a null vector. A direct computation gives

$$|A|^2 = -(A^0)^2 + (A^1)^2 = -1 + 1 = 0.$$

Similarly for

$$(1, -1, 0, 0), \quad (1, 0, 1, 0), \quad (1, \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0), \quad \text{etc.}$$

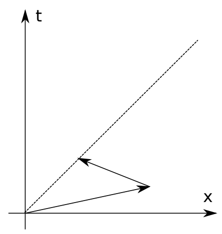


Inner (or scalar) product The scalar product of two 4-vectors A^a, B^b is defined by

$$A \cdot B = \eta_{ab} A^a B^b = -A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3.$$

Notice that as a consequence of this definition $|A|^2 = A \cdot A$.

Example: With the help of a sketch convince yourself that the sum of two timelike or spacelike vectors or the sum of a timelike and a spacelike vector can be null!



Manipulating tensors

The operation of contraction consists of summing over one upper and one lower index, thus turning a (k, l) tensor into a $(k-1, l-1)$ tensor:

$$S^{ab}{}_{c} = T^{adb}{}_{cd} \,.$$

One can check (**exercise**) that the object on the left hand side is a well-defined tensor. Note that it is only allowed to contract an upper index with a lower one, otherwise the result would *not* be a tensor. Also, in general the order of the indices matters, so one gets different tensors by contracting in different ways:

$$T^{abc}{}_{db} \neq T^{acb}{}_{db}$$

The metric and inverse metric can be used to raise and lower indices on tensors. That is, given a tensor $T^{ab}{}_{cd}$ we can use the metric to define new tensor with different positions of the indices:

$$\begin{aligned} T^{abm}{}_{d} &= \eta^{mc} \, T^{ab}{}_{cd} \,, \\ T_{m}{}^{b}{}_{cd} &= \eta_{ma} \, T^{ab}{}_{cd} \,, \\ T_{mn}{}^{pq} &= \eta_{ma} \, \eta_{nb} \, \eta^{pc} \, \eta^{qd} \, T^{ab}{}_{cd} \,, \end{aligned}$$

and so forth. Notice that raising or lowering does not change the position of the index relative to the other indices, and also that free indices (which are not summed over) must be the same on both sides of the equation, while dummy indices (which are summed over) only appear on one side. For example, we can turn vectors and dual vectors into each other by raising and lowering the indices:

$$V_a = \eta_{ab}\, V^b\,,$$
$$\omega^a = \eta^{ab}\, \omega_b\,.$$

Because the metric and inverse metric are inverses of each other, we can raise and lower simultaneously a pair of indices being contracted over:

$$A^a\, B_a = \eta^{bc}\, A_c\, \eta_{bd}\, B^d = \delta^c_d\, A_c\, B^d = A_d\, B^d\,.$$

Therefore, in spaces with a metric, we don't make a distinction between vectors and dual vectors.

Symmetries of tensors

A tensor is said to be symmetric in any of its indices if it is unchanged under the exchange of those indices. For example, if

$$S_{abc} = S_{bac}$$

we say that S_{abc} is symmetric in its first two indices, while if

$$S_{abc} = S_{acb} = S_{cab} = S_{bac} = S_{bca} = S_{cba}$$

we say that S_{abc} is symmetric in all of its indices. Similarly, a tensor is said to be anti-symmetric in any of its indices if it changes sign when those indices are exchanged. Hence,

$$A_{abc} = -A_{cba}$$

means that the tensor A_{abc} is anti-symmetric in its first and third indices. If a tensor is (anti-) symmetric in all of its indices, we refer to it as simply (anti-) symmetric (sometimes with the redundant modifier "completely"). Notice that it does not make sense to exchange upper and lower indices with each other, so for example Kronecker's delta δ^a_b is not symmetric nor anti-symmetric.

Given any tensor, we can symmetrise (or anti-symmetrise) any number of its lower or upper indices. To symmetrise,

$$T_{(a_1 a_2 \ldots a_n)b}{}^c = \frac{1}{n!}\left(T_{a_1 a_2 \ldots a_n b}{}^c + \text{sum over permutation of indices } a_1 \ldots a_n\right)\,,$$

while anti-symmetrisation comes with the alternating sum:

$$T_{[a_1 a_2 \ldots a_n]b}{}^c = \frac{1}{n!}\left(T_{a_1 a_2 \ldots a_n b}{}^c + \text{alternating sum over permutation of indices } a_1 \ldots a_n\right)\,,$$

By alternating sum we mean that permutations that are the result of an odd number of exchanges of indices are given a minus sign, for example,

$$T_{[abc]d} = \frac{1}{3!}\left(T_{abcd} - T_{acbd} + T_{cabd} - T_{bacd} + T_{bcad} - T_{cbad}\right)\,.$$

The standard notation is to use round/square brackets for symmetrisation/anti-symmetrisation. Sometimes we may want to (anti-)symmetrise indices that are not next to each other, in which case we use vertical bars to denote indices that are not included in the sum:

$$T_{(a|b|c)} = \frac{1}{2}\left(T_{abc} + T_{cba}\right) .$$

If we are contracting over a pair of indices that are symmetric/anti-symmetric on one tensor, only the symmetric/anti-symmetric part of the lower indices will contribute (**exercise**):

$$A^{(ab)}\, B_{ab} = A^{(ab)}\, B_{(ab)}\,, \qquad X^{[cd]}\, Y_{cd} = X^{[cd]}\, Y_{[cd]}\,,$$

regardless of the symmetry properties of B and Y respectively. For any *two* indices we can decompose a tensor into its symmetric and anti-symmetric parts,

$$T_{abcd} = T_{(ab)cd} + T_{[ab]cd}\,,$$

but this is not generally true for three or more indices:

$$T_{abce} \neq T_{(abc)d} + T_{[abc]d}\,,$$

because there are parts with mixed symmetry that are not specified by either the symmetric or anti-symmetric pieces. Note that according to the convention used here, a symmetric tensor S satisfies

$$S_{a_1 \ldots a_n} = S_{(a_1 \ldots a_n)}\,,$$

and likewise for anti-symmetric tensors.

If we think of $X^a{}_b$ as a matrix, we can sum the diagonal components to compute its trace and this makes sense. However, we will also want to compute the trace for a $(0, 2)$ tensor Y_{ab}, in which case we first have to raise an index ($Y^a{}_b = \eta^{ac} Y_{cb}$) and then contract:

$$Y = Y^a{}_a = \eta^{ab}\, Y_{ab}\,.$$

Note that the sum of the diagonal components of $Y^a{}_b$ is not the same as the sum of the diagonal components of Y_{ab}. Then, the trace of the Minkowski metric is given by

$$\eta^{ab}\, \eta_{ab} = \delta^a_a = 4\,.$$

Note that anti-symmetric $(0, 2)$ tensors are always tracelss (**check!**).

2.11 Proper time

In order to develop relativistic dynamics one requires the analogues of

$$v^a = \frac{\mathrm{d}x^a}{\mathrm{d}t}, \quad a^b = \frac{\mathrm{d}v^b}{\mathrm{d}t}, \quad F^c = \frac{\mathrm{d}p^c}{\mathrm{d}t},$$

etc. The problem is that in Special Relativity, $t = x^0$ is not a scalar, so that we cannot just carry $\mathrm{d}/\mathrm{d}t$ over to Special Relativity.

The closest thing to $\mathrm{d}t$ which is a scalar is the *proper time* interval $\mathrm{d}\tau$ defined by

$$\mathrm{d}\tau^2 \equiv -\frac{\mathrm{d}s^2}{c^2} = \mathrm{d}t^2 - \frac{\mathrm{d}x^2}{c^2} - \frac{\mathrm{d}y^2}{c^2} - \frac{\mathrm{d}z^2}{c^2}.$$

In the previous definition the minus sign is included so that $\mathrm{d}\tau$ and $\mathrm{d}t$ have the same sign! The name of proper time comes from the fact that a clock at rest with a moving particle —i.e. in the particle's rest frame where $\mathrm{d}x = \mathrm{d}y = \mathrm{d}z = 0$— has $\mathrm{d}\tau = \mathrm{d}\tau$ —i.e. it is equal to the time elapsed on the particle's clock.

We employ τ as the invariant measure of time for the particle.

2.12 4-velocity and 4-momentum

In order to express Newton's laws in Special Relativity in an invariant way, we need to express them in terms of 4-vectors.

4-velocity

The 4-velocity of a particle is defined as a unit tangent to its Worldline:

$$U^a = \frac{\mathrm{d}x^a}{\mathrm{d}\tau}, \quad U^i = \frac{\mathrm{d}x^i}{\mathrm{d}\tau}.$$

Notation: we will reserve the middle Latin indices $i, j, k, \ldots$ to denote the spatial components, so these indices range from 1 to 3. In what follows, for simplicity we set $c = 1$.

Remarks:

(1) From the definition of $\mathrm{d}\tau$ one finds that

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 = \mathrm{d}\bar{x} \cdot \mathrm{d}\bar{x} = \eta_{ab}\mathrm{d}x^a\mathrm{d}x^b$$

where $\mathrm{d}x^a = (\mathrm{d}t, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$ so that

$$U^a U_a = -1. \tag{2.24}$$

So that 4-velocity as defined has unit length.

(2) From $\mathrm{d}\tau^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$ one finds that

$$\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2 = 1 - \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 - \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2 = 1 - \underline{v}^2,$$

where $\underline{v}$ denotes the 3-velocity relative to the frame F and $\underline{v}^2 = \underline{v} \cdot \underline{v}$. Hence, one concludes that

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 - v^2}} = \gamma(v) \quad (c = 1).$$

Now, using

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma(v)v^1, \quad \text{etc}$$

one finds that

$$U^a = \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}, \frac{\mathrm{d}x}{\mathrm{d}\tau}, \frac{\mathrm{d}y}{\mathrm{d}\tau}, \frac{\mathrm{d}z}{\mathrm{d}\tau}\right) = \gamma(v)(1, v^1, v^2, v^3),$$

or in short

$$U^a = \gamma(v)(1, \underline{v}). \tag{2.25}$$

Note that the spatial part of U^a is essentially $\underline{v}$ (with a relativistic correction).

4-momentum

The 4-momentum is the natural analogue of the 3-momentum:

$$p^a = m_0 U^a,$$

where m_0 denotes the mass of the particle. From the definition it follows that

$$p^a p_a = m_0^2 U^a U_a = -m_0^2,$$

where it has been used that $U^a U_a = -1$. Also, using (2.25) one has

$$p^a = m_0\gamma(v)(1, \underline{v}). \tag{2.26}$$

It follows that the space part of (2.26) can be identified with the 3-momentum, where by analogy $m_0\gamma$ is called the *the moving mass*, or the apparent mass and m_0 is referred as the *rest mass*. From

$$m \equiv m_0\gamma(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}},$$

we see that $m \to \infty$ as $v \to c$, so it is impossible to accelerate a massive particle to the speed of light since its mass would effectively become infinite, thus requiring an infinite amount of energy.

Multiplying the time component of p^a by c, we can identify it with the energy

$$E = m_0 c^2 \gamma(v).$$

One reason for this identification comes from considering the limit for small v/c. For $v/c \ll 1$ one has

$$\begin{aligned} E &= m_0 c^2\gamma(v) = m_0 c^2 (1 - v^2/c^2)^{-1/2} \\ &\approx m_0 c^2 + \tfrac{1}{2} m_0 v^2 + O(v^4/c^2) \end{aligned}$$

where the binomial expansion has been used. Now, the second term is just the Newtonian kinetic energy ($\frac{1}{2}m_0 v^2$). The first term ($m_0 c^2$) is then interpreted as the rest mass energy. This is the famous equation

$$E_{rest} = m_0 c^2.$$

Note that since $c \approx 300{,}000\,\mathrm{km/s}$ is a very large number, E_{rest} is typically enormous.

From the previous discussion one can write ($c = 1$),

$$p^a = (E, \underline{p}), \tag{2.27}$$

with $\underline{p}$ the 3-momentum and E the energy. From (2.26) one concludes that

$$p^a p_a = (E, \underline{p}) \cdot (E, \underline{p}) = -E^2 + \underline{p} \cdot \underline{p} = -m_0^2\,,$$

and hence

$$E^2 - \underline{p} \cdot \underline{p} = m_0^2, \quad (c = 1).$$

4-acceleration

As one might expect, the 4-acceleration is the natural analogue of the 3-acceleration:

$$a^b = \frac{\mathrm{d}U^b}{\mathrm{d}\tau} = \frac{\mathrm{d}^2 x^b}{\mathrm{d}\tau^2},$$

where U^a is the 4-velocity previously defined. Observe that by differentiating the equation $U^a U_a = -1$, it follows that

$$a^b U_b = 0,$$

so that 4-acceleration and 4-velocity are found to be orthogonal.

2.13 Photons

The definition of 4-velocity given in the previous sections breaks down when applied to particles moving with the speed of light (photons) since for light rays one has $\mathrm{d}s^2 = -\mathrm{d}\tau^2 = 0$. In this case one may choose another parameter λ and define

$$k^a = \frac{\mathrm{d}x^a}{\mathrm{d}\lambda},$$

but again $k^a k_a = 0$ since k^a is null. This also implies that $p^a p_a = 0$ for photons as p^a is in the direction of U^a. Now, recalling that $p^a p_a = -m_0^2$, it follows that $m_0 = 0$ for photons. Hence, particles moving with the speed of light must be massless!

Consider a photon with 4-momentum $p^a = (E, \underline{p})$ defined relative to some frame F. As seen before $p^a p_a = 0$, so that one finds that

$$E^2 - p^2 = 0, \qquad \text{or } E = p.$$

Therefore, for photons the spatial 3-momentum and the energy are equal. In particular, if the photon moves along the x-direction one has that

$$p_x = E.$$

2.14 Doppler shift

Let F and F' be in standard configuration. Consider a photon of frequency ν moving in the x-direction relative to the frame F. Relative to the frame F' the energy of the photon may be obtained using a Lorentz transformation. For this recall that p^a is a 4-vector and its energy is given by its t-component. So, from

$$p^a = (E, p_x), \quad p_y = p_z = 0,$$

one obtains

$$E' = \gamma(E - v p_x), \quad (c = 1). \tag{2.28}$$

Also, recall that from Quantum Mechanics, a photon of frequency ν has energy given by $h\nu$ where h denotes Planck's constant:

$$h = 6.625 \times 10^{-34} Js.$$

Similarly, one has $E' = h\nu'$. Substituting in (2.28) one obtains

$$h\nu' = \frac{h\nu - vp_x}{\sqrt{1-v^2}}. \tag{2.29}$$

Furthermore, for such a photon $E = p_x$ so that substituting into (2.29):

$$h\nu' = \frac{h\nu - vh\nu}{\sqrt{1-v^2}},$$

from which we can conclude

$$\frac{\nu'}{\nu} = \frac{1-v}{\sqrt{1-v^2}} = \sqrt{\frac{1-v}{1+v}}.$$

Adding the constant c:

$$\frac{\nu'}{\nu} = \sqrt{\frac{1-v/c}{1+v/c}}. \tag{2.30}$$

This is the *relativistic Doppler shift formula.* Note that when $v/c \ll 1$, then using the binomial expansion in (2.30) one obtains

$$\frac{\nu'}{\nu} \approx 1 - v/c,$$

which is the usual (non-relativistic) formula for the Doppler shift.

Remark. The Doppler shift has been fundamental in Cosmology to establish the expansion of the Universe.

2.15 Relativistic dynamics

In Special Relativity Newton's laws become:

First law. Remains unchanged, except that the straight lines referred to are now worldlines in Minkowski spacetime.

Second law. One has

$$F^a = \frac{\mathrm{d}p^a}{\mathrm{d}\tau}.$$

Third law. On basis of very precise experiments of Particle Physics, this remains unchanged. That is, 4-momentum is conserved in collisions:

$$\sum_i p_i^a = \text{constant},$$

where the sum is over the particles involved in the collision.

Note. Due to constancy of the time component, the conservation of energy with rest mass is included in the balance!

2.15.1 Examples of relativistic collisions

This type of problems can be solved by equating components, squaring and then using further properties of p^a.

Example 1

Consider 2 particles with rest masses m_1 and m_2 both moving along collinearly with speeds u_1 and u_2. The particles collide and coalesce with the resulting particle moving in the same direction. The question is: what are the mass m and the speed u of the resulting particle?

Recall that $p^a = m\gamma(1, \underline{v})$ for a particle of 3-velocity $\underline{v}$. The initial 4-momenta are:

$$\begin{aligned} p_1^a &= m_1\gamma(u_1)(1, u_1, 0, 0), \\ p_2^a &= m_2\gamma(u_2)(1, u_2, 0, 0). \end{aligned}$$

The final 4-momentum is

$$p^a = m\gamma(u)(1, u, 0, 0).$$

The conservation of -momentum is expressed by

$$p^a = p_1^a + p_2^a. \tag{2.31}$$

Squaring

$$p^2 = p^a p_a = p_1^2 + p_2^2 + 2p_1 \cdot p_2. \tag{2.32}$$

However,

$$\begin{aligned} &p^2 = -m^2, \quad p_1^2 = -m_1^2, \quad p_2^2 = -m_2^2, \\ &p_1 \cdot p_2 = m_1 m_2 \gamma(u_1)\gamma(u_2)(-1 + u_1 u_2). \end{aligned}$$

Substituting in (2.32):

$$m = \sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \gamma(u_1)\gamma(u_2)(1 - u_1 u_2)}. \tag{2.33}$$

Taking space and t-components of 4-momenta in equation (2.31)

$$m\gamma(u)u = m_1\gamma(u_1)u_1 + m_2\gamma(u_2)u_2, \tag{2.34a}$$
$$m\gamma(u) = m_1\gamma(u_1) + m_2\gamma(u_2). \tag{2.34b}$$

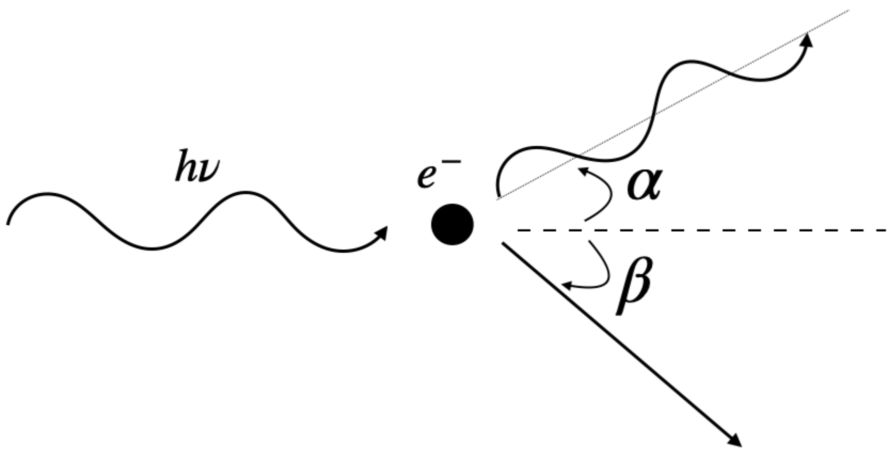
Dividing (2.34a) by (2.34b) one obtains

$$u = \frac{m_1\gamma(u_1)u_1 + m_2\gamma(u_2)u_2}{m_1\gamma(u_1) + m_2\gamma(u_2)}. \tag{2.35}$$

Remark. In the limit of $u_1 \ll c$ and $u_2 \ll c$ one has that $\gamma(u_1)$, $\gamma(u_2) \approx 1$ and that $(1 - u_1 u_2) \approx 1$ so that (2.33) and (2.35) yield

$$\begin{aligned} m &\approx m_1 + m_2, \\ u &\approx \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2}, \end{aligned}$$

which are the classical version of the result.



Example 2

Consider the collision (scattering) of a photon of frequency ν moving in the x-direction by an electron of mass m_e in a frame in which m_e is initially at rest. Assume that the subsequent motion remains in the xy plane. After the collision, the photon has frequency ν' and is moving at an angle α with the horizontal while the electron is moving at an angle β. Find the angles α and β in terms of frequencies before and after the collision, i.e., ν and ν', and the rest mass of the electron m_e.

Before the collision the 4-momenta of the photon and electron are given, respectively, by

$$p^a_{p_1} = h\nu(1, 1, 0, 0),$$
$$p^a_{e_1} = m_e\gamma(0)(1, 0, 0, 0), \quad \gamma(0) = 1.$$

After the collision we have that

$$p^a_{p_2} = h\nu'(1, \cos\alpha, \sin\alpha, 0),$$
$$p^a_{e_2} = m_e\gamma(v)(1, v\cos\beta, -v\sin\beta, 0),$$

where ν' is the new photon frequency and α, β are as given in the figure.

The conservation of 4-momentum gives:

$$p^a_{p_1} + p^a_{e_1} = p^a_{p_2} + p^a_{e_2}.$$

Squaring:

$$(p_{p_1} + p_{e_1} - p_{p_2}) \cdot (p_{p_1} + p_{e_1} - p_{p_2}) = p_{e_2} \cdot p_{e_2}. \tag{2.36}$$

But,

$$p^2_{e_1} = p^2_{e_2} = -m^2_e, \quad p^2_{p_1} = p^2_{p_2} = 0.$$

Substituting in (2.36) one obtains

$$p_{e_1} \cdot p_{p_1} - p_{e_1} \cdot p_{p_2} = p_{p_1} \cdot p_{p_2},$$

from where

$$-m_e h\nu + m_e h\nu' = h^2\nu\nu'(\cos\alpha - 1),$$

and

$$\sin^2\left(\tfrac{\alpha}{2}\right) = \frac{m_e c^2}{2h}\left(\frac{1}{\nu'} - \frac{1}{\nu}\right). \tag{2.37}$$

Similarly, to find β rewrite (2.36) as

$$(-p_{p_1} + p_{e_2} + p_{p_2}) \cdot (-p_{p_1} + p_{e_2} + p_{p_2}) = p_{e_1} \cdot p_{e_1}.$$

This example shows that the photon is deflected (or scattered) by and angle given by (2.37)

Chapter 3

Prelude to General Relativity

3.1 General remarks

At the time of the development of Special Relativity, physical interactions were supposed to be either gravitational or electromagnetic. Electromagnetism was already compatible with Special Relativity —i.e. invariant under Lorentz transformations. On the other hand, Newton's laws were not.

After the development of Special Relativity, what was needed was to construct a relativistic theory of gravity compatible with Special Relativity. The first attempts to construct such theory involved generalisations of Newton's laws of gravity. For example, Nordström developed a theory which was Lorentz invariant but which is incompatible with the observations —it does not produce light bending.

Einstein in 1915 succeeded in constructing a theory which is both Lorentz invariant and which s compatible with predictions. This theory is called *General Relativity.* In order to develop General Relativity, we will require some ingredients of *tensor calculus.* To understand why this mathematical tool is required, we take first a look at some of the principles that underlie the theory.

3.2 The Equivalence Principle

The Equivalence Principle amounts to the following two statements:

(1) The (equation of) motion of a (spherically symmetric) test particle (one whose own gravitational field may be neglected) in a gravitational field is independent of its mass and composition. The first verification of this statement is claimed to be Galileo's Pisa bell tower experiment —although this particular experiment probably never took place. More recent experiments like the one by Roll, Krotkov and Dicke (1964) have allowed to establish the equality to 1 part in 10^{11}.

(2) Matter (as well as every form of energy) is acted on by (and is itself a source of) gravitational field. In other words, gravity couples everything.

An immediate consequence of (2) is that it is not possible to eliminate the force of gravity in the same way that other forces may be eliminated, by for example, disconnecting power sources or by means of shielding as in the case of Faraday cages. The only other forces that behave in this way are the so-called *fictitious forces* (i.e. the centrifugal and Coriolis forces) which arise when non-inertial frames of reference are employed. The important point about these forces is that like gravity, they are proportional to the mass of the particle. This led Einstein to suspect that these and the gravitational forces should enter the theory in the same way.

To get a better feeling for this, recall that the only way one can eliminate the force of gravity is by choosing a freely falling frame —i.e. a comoving frame with the freely falling particle. This is can be visualised in the thought experiment (Gedankenexperiment) — sometimes referred to as the *lift experiment*.

The experiment suggests that there are no local experiments which distinguish non-rotating free fall in gravitational field from a uniform motion in a space free from gravitational fields. By local, here its is understood that the experiment is performed in a small region such that the variation of the gravitational field is negligible (observationally). This is another way of expressing the Equivalence Principle (all particles fall in the same way). In this sense, Special Relativity is regained locally, in the sense that the laws of Physics in a freely falling frame are compatible with Special Relativity. Alternatively, one can say that spacetime is locally Minkowskian. Furthermore, for a global theory in the presence of gravitation (i.e. GR), the geometry of spacetime must be such that it is locally Minkowskian. The natural tool to express and implement these ideas is the so-called *tensor calculus*.

3.3 Summary

In presence of gravitational fields there exist, in small regions (locally), preferred inertial frames (i.e. the non-rotating free falling frames) in which the special relativistic results hold. On a large scale, on the other hand, there are no such preferred frames, and hence one needs to treat all large scale reference frames on the same footing. This suggests that the laws of nature should be formulated in such a way that they are invariant under arbitrary transformations of coordinates (i.e. reference frames), and not just the Lorentz transformations as was the case of Special relativity.

Interpreted physically, this is called the *General Principle of Relativity* as opposed to the Special Principle of Relativity according to which laws of nature have the same form in inertial frames.

Interpreted mathematically, it is called the principle of *General Covariance* —the equations of Physics should have tensorial form.

Chapter 4

Differential Geometry and tensor calculus

In describing spacetime we wish our equations to be valid for any coordinates. Tensorial equations satisfy this property —hence their significance.

4.1 Manifolds and coordinates

Manifolds are fundamental concepts in mathematics and physics. We are used to the properties of n-dimensional Euclidean space, $\mathbb{R}^n$, the set of n-tuples $(x^1, \ldots, x^n)$ often equipped with the positive-definite metric δ_{ij}. There is the well-known theory of analysis in $\mathbb{R}^n$, e.g., differentiation, integration, etc.. However, there are other spaces, e.g., spheres, torii, which are "curved" or have topology, where we would like to perform analogous operations. The notion of manifold addresses this issue. A manifold is a space that can be curved and/or can have a complicated topology but locally it looks like $\mathbb{R}^n$. The entire manifold is constructed by sewing together these local regions. For example, $\mathbb{R}^n$, an n-dimensional sphere S^n or an n-torus T^n are manifolds. While a single cone is another example of a manifold, two cones intersecting at the vertex are not a manifold because at the vertex the space does not look like Euclidean space.

To formalise the notions of "looking locally like $\mathbb{R}^n$" and "smoothly sewn together" we need some preliminary definitions. Given two sets M and N, a map $\phi : M \to N$ is a relationship that assigns, to each element of M, exactly one element of N. Hence, a map is just a generalisation of the concept of a function. Given two maps $\phi : A \to B$ and $\psi : B \to C$, we define the composition $\psi \circ \phi : A \to C$ by the operation $(\psi \circ \phi)(a) = \psi(\phi(a))$, so $a \in A$, $\phi(a) \in B$ and hence $(\psi \circ \phi)(a) \in C$. The order in which the maps are written is such that the one on the right acts first.

A map is called one-to-one (or injective) if each element of N has at most one element of M mapped into it, and onto (or surjective) if each element of N has at least one element of M mapped into it. For example, consider functions $\phi : \mathbb{R} \to \mathbb{R}$. Then $\phi(x) = e^x$ is one-to-one but not onto; $\phi(x) = x^3 - x$ is onto, but not one-to-one; $\phi(x) = x^3$ is both; and $\phi(x) = x^2$ is neither.

The set M is known as the domain of the map ϕ, and the set of elements in N that M gets mapped into is called the image of ϕ. For any subset $U \subset N$, the set of elements of M that get mapped to U is called the preimage of U under ϕ or $\phi^{-1}(U)$. A map that is both one-to-one and onto is known as invertible (or bijective). In this case we can define the inverse map $\phi^{-1} : N \to M$ by $(\phi^{-1} \circ \phi)(a) = a$.

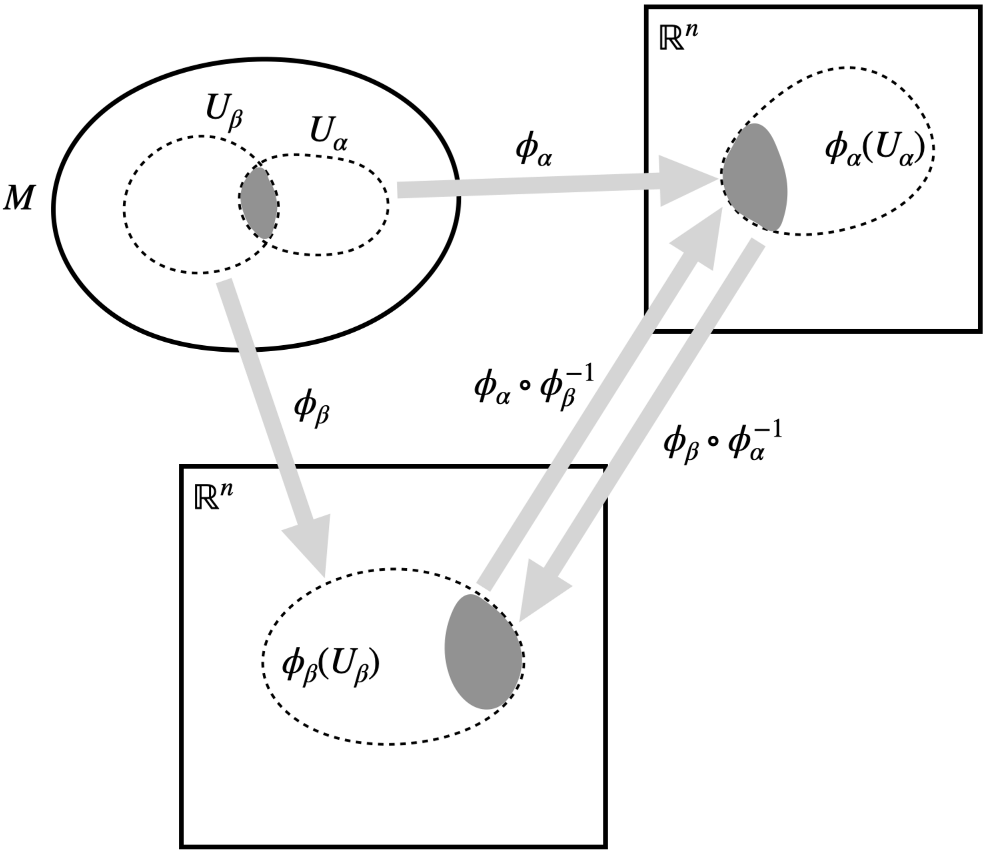


Figure 4.1: Overlapping coordiante charts. The maps are only defined on the shaded regions and they should be smooth there.

Consider maps between general Euclidean spaces $\phi : \mathbb{R}^m \to \mathbb{R}^n$. A map from $\mathbb{R}^m$ to $\mathbb{R}^n$ takes an m-tuple $(x^1, \dots, x^m)$ to an n-tuple $(y^1, \dots, y^n)$ and can therefore be thought of as a collection of n functions ϕ^i of m variables:

$$\begin{aligned} y^1 &= \phi^1(x^1, \dots, x^m) \\ y^2 &= \phi^2(x^1, \dots, x^m) \\ &\vdots \\ y^n &= \phi^n(x^1, \dots, x^m) \end{aligned} \tag{4.1}$$

We will refer to any of these functions as C^p if its pth derivative exists and is continuous, and refer to the entire map $\phi : \mathbb{R}^m \to \mathbb{R}^n$ as C^p if each of its component functions is at least C^p. The C^0 map is continuous but not differentiable, while a C^∞ map is continuous and it can be differentiated infinitely many times; such a map is also called smooth.

We will call two sets M and N diffeomorphic if there exists a C^∞ map $\phi : M \to N$ with a C^∞ inverse $\phi^{-1} : N \to M$; the map ϕ is then called a diffeomorphism. This is the best notion we have to say that two spaces are "the same".

Now we can proceed to rigorously define a manifold. Consider first an open ball, namely the set of all points x in $\mathbb{R}^n$ such that $|x - y| < r$ for some fixed $y \in \mathbb{R}^n$ and $r \in \mathbb{R}$, where $|x - y| = [\sum_{i=1}^n (x^i - y^i)^2]^{1/2}$ is the usual Euclidean norm. An open set in $\mathbb{R}^n$ is a set constructed from an arbitrary union of open balls. A chart or coordinate

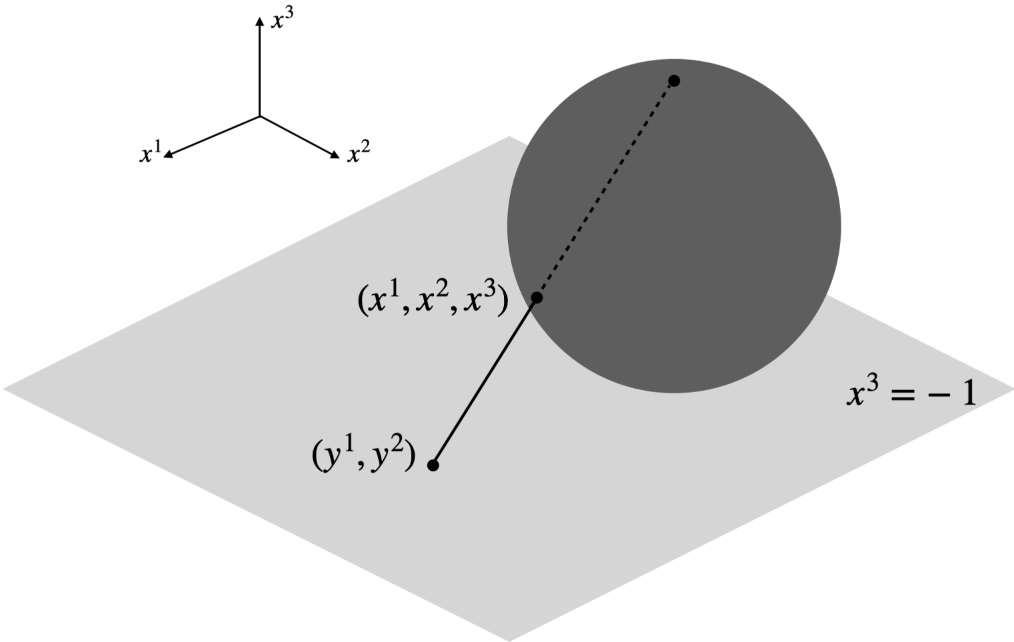


Figure 4.2: Stereographic coordinates on the S^2 defined by projecting from the north pole down to the $x^3 = -1$ plane. This coordinate chart covers the whole sphere except for the north pole.

system consists of a subset U of a set M along with a one-to-one map $\phi : U \to \mathbb{R}^n$ such that the image $\phi(U)$ is open in $\mathbb{R}^n$. We then say that U is an open set in M. A C^∞ atlas is an indexed collection of charts $[(U_\alpha, \phi_\alpha)]$ that satisfies two conditions:

1. The union of the U_α is equal to M, that is, the U_α cover M.

2. The charts are smoothy sewn together. More precisely, if two charts overlap, $U_\alpha \cap U_\beta \neq \emptyset$, then the map $(\phi_\alpha \circ \phi_\beta^{-1})$ takes points in $\phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$ onto an open set $\phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$, and all these maps must be C^∞ where they are defined.

See Fig. 4.1. So a chart is what we normally think of as a coordinate system on some open set, and an atlas is a collection of charts that are smoothly related on their overlaps. Then a C^∞ n-dimensional manifold is simply a set M along with a maximal atlas, one that contains every possible compatible chart. We can replace C^∞ by C^p in the definition but in our applications we will always consider smooth manifolds. The requirement that the atlas be maximal is so that two equivalent spaces equipped with different atlases do not count as different manifolds.

A non-trivial example of a manifold is the two-dimensional sphere, S^2. Let's take the S^2 of unit radius to be the set of points in $\mathbb{R}^3$ defined by $(x^1)^2 + (x^2)^2 + (x^3)^2 = 1$. We can construct a chart from the open set U_1 defined by the sphere minus the north pole via the stereographic projection (see Fig. 4.2: draw a straight line from the north pole to the plane defined by $x^3 = -1$, and assign to the point on the S^2 intercepted by the line the Cartesian coordinates (y^1, y^2) of the appropriate point on the plane. Explicitly, the map is given by

$$\phi_1(x^1, x^2, x^3) \equiv (y^1, y^2) = \left(\frac{2\,x^1}{1 - x^3}, \frac{2\,x^2}{1 - x^3} \right) .$$

Another chart (U_2, ϕ_2) can be obtained by projecting from the south pole to the plane $x^3 = +1$. The resulting coordinates cover the sphere minus the south pole, and are given by:

$$\phi_2(x^1, x^2, x^3) \equiv (z^1, z^2) = \left(\frac{2\,x^1}{1+x^3}, \frac{2\,x^2}{1+x^3} \right) .$$

Together, these two charts cover the entire manifold, and they overlap in the region $-1 < x^3 < 1$. One can check that the composition $\phi_2 \circ \phi_1^{-1}$ is given by

$$z^i = \frac{4\,y^i}{\sqrt{(y^1)^2 + (y^2)^2}} , \quad i = 1, 2 ,$$

which is C^∞ in the overlap region.

Consider the maps $f \ : \ \mathbb{R}^m \to \mathbb{R}^n$ and $g \ : \ \mathbb{R}^n \to \mathbb{R}^l$, and the composition map $(g \circ f) \ : \ \mathbb{R}^m \to \mathbb{R}^l$. We can label points on each space in terms of the usual Cartesian coordinates: x^a on $\mathbb{R}^m$, y^b on $\mathbb{R}^n$ and z^c on $\mathbb{R}^l$, where the indices range over the appropriate values. The chain rule relates the partial derivatives of the composition to the partial derivatives of the individual maps:

$$\frac{\partial}{\partial x^a}(g \circ f)^c = \sum_b \frac{\partial f^b}{\partial x^a} \frac{\partial g^c}{\partial y^b} .$$

This is usually abbreviated to

$$\frac{\partial}{\partial x^a} = \sum_b \frac{\partial y^b}{\partial x^a} \frac{\partial}{\partial y^b} . \tag{4.2}$$

When $m = n$, the determinant of the matrix $\frac{\partial y^b}{\partial x^a}$ is called Jacobian of the map, and the map is invertible whenever the Jacobian is non-zero.

4.2 Vectors

To construct the tangent space T_p using objects that are intrinsic to M we proceed as follows. Let $\mathcal{F}$ be the space of all smooth functions on M, i.e., C^∞ maps $f \ : \ M \to \mathbb{R}$. Each curve through p defines an operator on this space, namely the directional derivative, which maps $f \to \frac{df}{d\lambda}$ at p. Then the tangent space T_p can be identified with the space of directional derivative operators along curves through p. To see that this is indeed the case, note that two operators $\frac{d}{d\lambda}$ and $\frac{d}{d\eta}$ representing derivatives along two curves $x^a(\lambda)$ and $x^a(\eta)$ through p can be added and scaled by real numbers to give another operator $a\frac{d}{d\lambda} + b\frac{d}{d\eta}$. This new operator clearly acts linearly on functions and it can also be shown to satisfy the Leibniz rule. Therefore, the set of directional derivatives forms a vector space.

To identify the vector space of directional derivatives with the tangent space T_p we need to show the directional derivatives form a suitable basis for this space. To construct such a basis, consider a coordinate chart with coordinates x^a. An obvious set of n-directional derivatives at p are the partial derivatives ∂_a at this point. Note that this is the definition of partial derivative with respect to x^a: the directional derivative along a curve defined by $x^b =$ constant for all $b \neq a$, parametrised by x^a itself. We are now going to show that the partial derivative operators $\{\partial_a\}$ at p form a basis for the tangent space T_p. To do this, we are going to show that any directional derivative can be decomposed

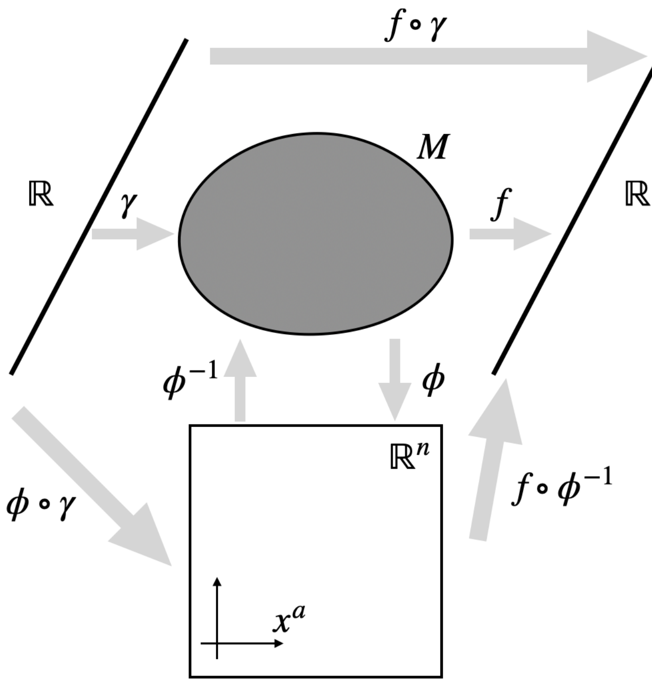


Figure 4.3: Decomposing the tangent vector to a curve $\gamma : \mathbb{R} \to M$ in terms of partial derivatives with respect to local coordinates on M.

into a linear combination of partial derivatives. Consider an n-dimensional manifold M, a coordinate chart $\phi : M \to \mathbb{R}^n$, a curve $\gamma : \mathbb{R} \to M$, and a function $f : M \to \mathbb{R}$. This leads to the tangle of maps shown in Fig. 4.3. If λ is the parameter along γ, we want to express the vector/operator $\frac{d}{d\lambda}$ in terms of partial derivatives ∂_a. Using the chain rule, we have

$$\begin{aligned} \frac{d}{d\lambda} f &= \frac{d}{d\lambda}(f \circ \gamma) \\ &= \frac{d}{d\lambda}[(f \circ \phi^{-1}) \circ (\phi \circ \gamma)] \\ &= \frac{d(\phi \circ \gamma)^a}{d\lambda} \frac{\partial (f \circ \phi^{-1})}{\partial x^a} \\ &= \frac{dx^a}{d\lambda} \partial_a f \,. \end{aligned} \tag{4.3}$$

The first line simply takes the informal expression on the left hand side and writes it as an honest derivative of the function $(f \circ \gamma) : \mathbb{R} \to \mathbb{R}$. The second line just comes from the definition of inverse map ϕ^{-1}. The third line is just the formal chain rule and the last line is a return to the informal notation of the start. Since the function f is arbitrary, we have

$$\frac{d}{d\lambda} = \frac{dx^a}{d\lambda} \partial_a \,.$$

Thus, the partial derivatives $\{\partial_a\}$ do indeed represent a good basis for the vector space of directional derivatives, and hence we can identify the latter with the tangent space.

This particular basis, i.e., $\hat{e}_{(a)} = \partial_a$, is known as a coordinate basis for T_p; it is the formalisation of the notion of setting up basis vectors to point along the coordinate axes. In general, we do not have to limit ourselves to coordinate bases when we consider tangent vectors. For example, the coordinate basis vectors are typically not normalised to unity nor orthogonal to each other. In fact, on a curved manifold, the coordinate basis will never be orthogonal throughout the neighbourhood of a point where the curvature does not vanish. One can define non-coordinate orthonormal bases by giving their components in a coordinate basis but we will not use this in these lectures.

One of the advantages of the abstract point of view that we have taken regarding vectors is that now the transformation law under changes of coordinates is immediate. Since the basis vectors are $\hat{e}_{(a)} = \partial_a$, the basis vectors in some new coordinate system $x^{a'}$ are simply given by the chain rule (4.2):

$$\partial_{a'} = \frac{\partial x^a}{\partial x^{a'}} \partial_a \,.$$

Just as in flat space, we can get the transformation law for the components of a vector V by demanding that $V = V^a \partial_a$ is unchanged by a change of basis:

$$\begin{aligned} V^a \partial_a &= V^{a'} \partial_{a'} \\ &= V^{a'} \frac{\partial x^a}{\partial x^{a'}} \partial_a \,, \end{aligned} \tag{4.4}$$

and hence, since the matrix $\frac{\partial x^{a'}}{\partial x^a}$ is the inverse of the matrix $\frac{\partial x^a}{\partial x^{a'}}$, we have

$$V^{a'} = \frac{\partial x^{a'}}{\partial x^a} V^a \,. \tag{4.5}$$

Since the basis vectors are usually not written explicitly, the rule (4.5) for transforming components of vectors is what we call "vector transformation law". Note that this law is compatible with the transformation of vector components in Special Relativity under Lorentz transformations, $V^{a'} = L^{a'}{}_a V^a$ since Lorentz transformations are just a very special kind of coordinate transformations, namely $x^{a'} = L^{a'}{}_a x^a$. However, (4.5) is completely general and it determines the transformation of vectors under arbitrary changes of coordinates.

Since a vector at a point can be thought of as directional derivative along a path through that point, a vector field defines a map from smooth functions to smooth functions all over the manifold by taking a derivative at each point. Given two vector fields X and Y, we can define the commutator $[X, Y]$ by its action on an arbitrary function $f(x^a)$:

$$[X, Y](f) \equiv X(Y(f)) - Y(X(f)) \,. \tag{4.6}$$

Clearly this operator is independent of the coordinates. Moreover, the commutator of two vector fields is itself a vector field: if f and g are functions and a and b are real numbers, the commutator is linear:

$$[X, Y](af + bg) = a[X, Y](f) + b[X, Y](g) \,,$$

and it obeys the Leibniz rule,

$$[X, Y](fg) = f[X, Y](g) + g[X, Y](f) \,.$$

Exercise: show that the components of the vector field $[X, Y]^a$ are given by

$$[X, Y]^a = X^b \partial_b Y^a - Y^c \partial_c X^a \,.$$

Exercise: using the result above show that the components of $[X, Y]^a$ transforms as a vector under general coordinate transformations.
Remark. The commutator is a special case of the Lie derivative.

4.3 Tensors

Having defined vectors on general manifolds, we can now consider dual vectors (one-forms). Once again, the co-tangent space T_p^* can be thought of as the set of linear maps $\omega : T_p \to \mathbb{R}$. The canonical example of a one-form is the gradient of a function f, denoted by df. Its action on a vector $\frac{d}{d\lambda}$ is the directional derivative of the function:

$$df\left(\frac{d}{d\lambda}\right) = \frac{df}{d\lambda}\,. \tag{4.7}$$

Like in the case of a vector, a one-form exists only at the point where it is defined and it does not depend on information at other points in M.

Just as the partial derivatives along the coordinate axes provide a natural basis for the tangent space, the gradients of the coordinate functions x^a provide a natural basis for the co-tangent space. In flat space we constructed a basis for T_p^* by demanding that $\hat{\theta}^{(a)}(\hat{e}_{(b)}) = \delta^a_b$. On an arbitrary manifold M we can do the same and (4.7) leads to

$$dx^a(\partial_b) = \frac{\partial x^a}{\partial x^b} = \delta^a_b\,. \tag{4.8}$$

Therefore the gradients $\{dx^a\}$ are an appropriate basis of one-forms, and hence an arbitrary one-form can be expanded into components as $\omega = \omega_a\, dx^a$.

The transformation rules of basis dual vectors and components follow from the usual procedure. For the basis one-forms, we get

$$dx^{a'} = \frac{\partial x^{a'}}{\partial x^a}\, dx^a\,, \tag{4.9}$$

and for the components,

$$\omega_{a'} = \frac{\partial x^a}{\partial x^{a'}}\, \omega_a\,. \tag{4.10}$$

We will usually write the components ω_a we we refer to a one-form ω.

Just as in flat space, a $(k.l)$ tensor is a multilinear map from k dual vectors and l vectors to $\mathbb{R}$. Its components in a coordinate basis can be obtained by acting the tensor on the basis of one-forms and vectors,

$$T^{a_1\ldots a_k}{}_{b_1\ldots b_l} = T(dx^{a_1}, \ldots, dx^{a_k}, \partial_{b_1}, \ldots, \partial_{b_l})\,. \tag{4.11}$$

This is equivalent to the expansion

$$T = T^{a_1\ldots a_k}{}_{b_1\ldots b_l}\, \partial_{a_1} \otimes \cdots \otimes \partial_{a_k} \otimes dx^{b_1} \otimes \cdots \otimes dx^{b_l}\,.$$

The transformation law for general tensors under general changes of coordinates follows exactly the same pattern as in flat space, now replacing the Lorentz transformation matrix used in flat space with the Jacobian of the general coordinate transformation:

$$T^{a'_1\ldots a'_k}{}_{b'_1\ldots b'_l} = \frac{\partial x^{a'_1}}{\partial x^{a_1}} \cdots \frac{\partial x^{a'_k}}{\partial x^{a_k}} \frac{\partial x^{b_1}}{\partial x^{b'_1}} \cdots \frac{\partial x^{b_l}}{\partial x^{b'_l}}\, T^{a_1\ldots a_k}{}_{b_1\ldots b_l}\,. \tag{4.12}$$

It is often easier (but entirely equivalent) to transform a tensor under coordinate transformations by taking the identity of basis vectors and one-forms as partial derivatives and gradients at face value, and simply substituting in the coordinate transformation.

Example: Consider a symmetric $(0,2)$ tensor S on a two-dimensional manifold whose components in a coordinate system $(x^1 = x, x^2 = y)$ are given by:

$$S_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & x^2 \end{pmatrix} .$$

This can be equivalently written as

$$S = S_{ab}\, dx^a \otimes dx^b = dx \otimes dx + x^2\, dy \otimes dy\,, \tag{4.13}$$

where in the last equality we suppress the tensor product symbols for brevity (as this is common practice!). Consider the coordinate transformation:

$$x' = \frac{2\,x}{y}\,, \quad y' = \frac{y}{2}\,,$$

(valid for example when $x > 0$ and $y > 0$). This coordinate transformation can be straightforwardly inverted:

$$x = x'\,y'\,, \quad y = 2\,y'\,. \tag{4.14}$$

To calculate the components of S in the new coordinate system, $S_{a'b'}$, we could easily compute the matrix $\frac{\partial x^a}{\partial x^{a'}}$ (and its inverse $\frac{\partial x^{a'}}{\partial x^a}$ if needed) and apply the general formula for the tensor transformation law (4.12). Instead, we will use the fact that we compute the derivatives of the old coordinates in terms of the new ones using (4.14) to express dx^a in terms of $dx^{a'}$:

$$\begin{aligned} dx &= y'\,dx' + x'\,dy' \\ dy &= 2\,dy'\,. \end{aligned}$$

Plugging these expressions into (4.13) and remembering that the tensor products do not commute ($dx' \otimes dy' \neq dy' \otimes dx'$), we obtain,

$$\begin{aligned} S &= dx \otimes + x^2\, dy \otimes dy \\ &= (y'\,dx' + x'\,dy') \otimes (y'\,dx' + x'\,dy') + (x'y')^2\,(2\,dy') \otimes (2\,dy') \\ &= (y')^2 dx' \otimes dx' + x'\,y'(dx' \otimes dy' + dy' \otimes dx') + [(x')^2 + 4\,(x'\,y')^2] dy' \otimes dy'\,, \end{aligned}$$

which is equivalent to

$$S_{a'b'} = \begin{pmatrix} (y')^2 & x'\,y' \\ x'\,y' & (x')^2 + 4\,(x'\,y')^2 \end{pmatrix} .$$

Most tensor operations defined in flat space, e.g., contraction, symmetrisation, etc., are unaltered on a general manifold. However, the partial derivative of a general tensor is not, in general, a new tensor. The gradient, which is the partial derivative of a scalar is an honest $(0,1)$ tensor, but the partial derivative of a higher rank tensor is not a tensor. To see this, for example, let's consider the transformation of $\partial_a W_b$, where W_a is a $(0,1)$ tensor, under a general coordinate transformation:

$$\begin{aligned} \frac{\partial}{\partial x^{a'}} W_{b'} &= \frac{\partial x^a}{\partial x^{a'}} \frac{\partial}{\partial x^a} \left(\frac{\partial x^b}{\partial x^{b'}}\, W_b \right) \\ &= \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}} \left(\frac{\partial}{\partial x^a}\, W_b \right) + W_b \frac{\partial x^a}{\partial x^{a'}} \frac{\partial}{\partial x^a} \left(\frac{\partial x^b}{\partial x^{b'}} \right) . \end{aligned} \tag{4.15}$$

The first term on the right hand side is the expected one for the transformation law of a $(0,2)$ tensor but the second one spoils the correct transformation.

4.4 The metric

The metric tensor in a general curved space is denoted by the symbol g_{ab} (while η_{ab} is specifically reserved for the Minkowski metric). The metric tensor g_{ab} is a symmetric $(0, 2)$ tensor and, at least in these lectures, we will take it to be non-degenerate, that is, its determinant $g \equiv |g_{ab}|$ does not vanish. In this case, we can define the inverse metric g^{ab} via

$$g^{ab}\, g_{bc} = g_{cd}\, g^{da} = \delta^a_c\,.$$

Since g_{ab} is symmetric, the inverse metric g^{ab} is also symmetric. Just as in special relativity, the metric and its inverse can be used to raise and lower indices of tensors.

Here we outline some of the reasons why the metric is such an important object:

1. The metric provides a notion of "past" and "future".

2. The metric allows to compute the length of paths and proper time.

3. The metric determines the "shortest" distance between two points (and therefore the motion of test particles).

4. In general relativity, the metric replaces the Newtonian gravitational field ϕ.

5. The metric provides a notion of locally inertial frames and therefore a sense of "no rotation".

6. The metric determines causality, by defining the speed of light faster than which no signal can travel.

7. The metric replaces the usual Euclidean three-dimensional dot product.

In our discussion of proper time and special relativity we introduced the line element $ds^2 = \eta_{ab}dx^a dx^b$, which we used to calculate the length of the path. Now we now that dx^a is really a basis of dual vectors. In a general curved manifold, the line element is given by

$$ds^2 = g_{ab}(x)\, dx^a\, dx^b\,. \tag{4.16}$$

For example, the line element of three-dimensional Euclidean space in Cartesian coordinates is

$$ds^2 = (dx)^2 + (dy)^2 + (dz)^2\,.$$

In spherical coordinates,

$$\begin{aligned} x &= r\, \sin\theta\, \cos\theta\,, \\ y &= r\, \sin\theta\, \sin\theta\,, \\ z &= r\, \cos\theta\,, \end{aligned}$$

the line element becomes (**exercise**),

$$ds^2 = dr^2 + r^2\, d\theta^2 + \sin^2\theta\, d\phi^2\,.$$

It can be shown that, at some given point p on a manifold M, the metric g_{ab} can always be put into its canonical form, where its components are

$$g_{ab} = \mathrm{diag}(-1, -1, \ldots, -1, +1, +1, \ldots, +1, 0, 0, \ldots, 0)\,.$$

Furthermore, $\partial_c g_{ab}|_p = 0$ but $\partial_c \partial_d g_{ab}|_p \neq 0$. The signature of the metric is the number of both positive and negative eigenvalues. If all signs are positive, the metric is called Euclidean or Riemannian (or just positive definite), while if there is a single minus it is called Lorentzian or pseudo-Riemannian, and any metric with some $+1$'s and some -1's is called indefinite.

With the definition (4.16) of a general metric g_{ab} on a curved space, we can straightforwardly generalise many of the notions we had in Minkowski space:

Norm of a vector: Given a vector V^a, the norm is defined via

$$|V|^2 \equiv g_{ab} V^a V^b.$$

If $|V|^2 > 0$ (or $|V|^2 < 0$) for all vectors V^a, the metric is called positive definite (or negative definite) —this is the Riemannian case. Otherwise it is called indefinite —this includes the Lorentzian case.

Inner (scalar) product between two vectors: Given two arbitrary vectors A^a and B^a, their inner (scalar) product is defined as

$$A \cdot B \equiv g_{ab} A^a B^b.$$

If $g_{ab} A^a B^b = 0$, then A^a and B^b are said to be orthogonal.

Null vectors: For indefinite metrics, null vectors are those that have zero norm, i.e., they are orthogonal to themselves: For indefinite metrics there are vectors that are orthogonal to themselves. That is,

$$g_{ab} A^a A^b = 0.$$

Note for indefinite metrics, note that this does *not* imply that A^a is the zero vector ($A^a = 0$).

4.5 Covariant derivatives

In flat space in inertial coordinates, the partial derivative operator ∂_a is a map from (k, l) tensor fields to $(k, l+1)$ tensor fields which acts linearly on its arguments and obeys the Leibniz rule on tensor products. However, we have seen in (4.15) that on a general curved manifold, it is no longer true that the partial derivative operator ∂_a operator acting on a tensor produces another tensor. Therefore, we need to define a new derivative operator, called covariant derivative and denoted by the symbol ∇, which is independent of the coordinates and that maps (k, l) tensor fields to $(k, l+1)$ tensor fields. Given two arbitrary tensor fields T and S, then we demand that ∇ obeys:

1. Linearity: $\nabla(T + S) = \nabla T + \nabla S$.
2. Leibniz rule: $\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$.

It can be shown that if ∇ has to satisfy the Leibniz rule, then it can always be written as the partial derivative plus some linear transformation. For the case of a vector field V^a, this implies

$$\nabla_a V^b = \partial_a V^b + \Gamma^b{}_{ac} V^c \,, \tag{4.17}$$

where the $\Gamma^b{}_{ac}$'s are called connection coefficients. We can determine the transformation rule for the connection coefficients by demanding that the covariant derivative of a vector (4.17) transforms as a $(1,1)$ tensor under coordinate transformations, namely

$$\nabla_{a'} V^{b'} = \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^{b'}}{\partial x^b} \nabla_a V^b .$$

Expanding the left hand side,

$$\begin{aligned}
\nabla_{a'} V^{b'} &= \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^{b'}}{\partial x^b} \nabla_a V^b \\
&= \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^{b'}}{\partial x^b} \partial_a V^b + \frac{\partial x^a}{\partial x^{a'}} V^b \frac{\partial}{\partial x^a} \left(\frac{\partial x^{b'}}{\partial x^b} \right) + \Gamma^{b'}{}_{a'c'} \frac{\partial x^{c'}}{\partial x^c} V^c .
\end{aligned}$$

Expanding now the right hand side:

$$\frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^{b'}}{\partial x^b} \nabla_a V^b = \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^{b'}}{\partial x^b} \partial_a V^b + \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^{b'}}{\partial x^b} \Gamma^b{}_{ac} V^c .$$

The last two expressions have to be equated; the first terms in each side are the same and hence they cancel, so we are left with

$$\Gamma^{b'}{}_{a'c'} \frac{\partial x^{c'}}{\partial x^c} V^c + \frac{\partial x^a}{\partial x^{a'}} V^c \frac{\partial}{\partial x^a} \left(\frac{\partial x^{b'}}{\partial x^c} \right) = \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^{b'}}{\partial x^b} \Gamma^b{}_{ac} V^c .$$

Notice that in the second term on the left hand side of this equation we have changed the dummy index $b \to c$. Multiplying this equation by $\frac{\partial x^c}{\partial x^{d'}}$ and relabelling $d' \to c'$ we find

$$\Gamma^{b'}{}_{a'c'} = \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^c}{\partial x^{c'}} \frac{\partial x^{b'}}{\partial x^b} \Gamma^b{}_{ac} - \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^c}{\partial x^{c'}} \frac{\partial^2 x^{b'}}{\partial x^a \partial x^c} . \tag{4.18}$$

This is not a tensor transformation law since the second term on the right hand side spoils it; therefore, it is clear that the connection coefficients are not the components of a tensor. They are precisely constructed to be non-tensorial but in way such that the combination (4.17) transforms as a tensor.

Equation (4.18) does not determine the connection coefficients uniquely. To further constraint the connection, we impose that the covariant derivative satisfies the following two properties, in addition to the previous ones:

3. Commutes with contractions: $\nabla_a (T^c{}_{cb}) = (\nabla T)_a{}^c{}_{cb}$.

4. Reduces to the partial derivative when acting on scalars: $\nabla_a \phi = \partial_a \phi$.

The first property is equivalent to demand that the Kronecker delta (the identity map) is covariantly constant: $\nabla_a \delta^b_c = 0$, which resonable to impose since the components of δ^a_b are constants (zeros and ones).

Let's see what these new properties imply. Consider an arbitrary one-form ω_a and an arbitrary vector field V^a; we can compute the covariant derivative of the scalar defined by $\omega_a V^a$ to obtain:

$$\begin{aligned}
\nabla_a (\omega_b V^b) &= (\nabla_a \omega_b) V^b + \omega_b (\nabla_a V^b) \\
&= (\nabla_a \omega_b) V^b + \omega_b (\partial_a V^b + \Gamma^b{}_{ac} V^c) .
\end{aligned}$$

But

$$\begin{aligned}\nabla_a(\omega_b V^b) &= \partial_a(\omega_b V^b)\\ &= V^b\partial_a\omega_b + \omega_b\partial_a V^b\end{aligned}$$

Equating these two expressions, re-labelling the dummy indices $b \to c$ and $c \to b$ in the term with the connection in the first expression and recalling that V^a is arbitrary, we obtain the formula for the covariant derivative of a one-form

$$\nabla_a\omega_b = \partial_a\omega_b - \Gamma^c{}_{ab}\,\omega_c\,. \tag{4.19}$$

It is now straightforward to determine the formula for the covariant derivative acting on an arbitrary (k,l) tensor. We find,

$$\begin{aligned}\nabla_c T^{a_1\ldots a_k}{}_{b_1\ldots b_l} = \partial_c T^{a_1\ldots a_k}{}_{b_1\ldots b_l}\\ &+ \Gamma^{a_1}{}_{cd}\,T^{da_2\ldots a_k}{}_{b_1\ldots b_l} + \cdots + \Gamma^{a_k}{}_{cd}\,T^{a_1\ldots a_{k-1}d}{}_{b_1\ldots b_l}\\ &- \Gamma^d{}_{ab_1}\,T^{a_1\ldots a_k}{}_{db_2\ldots b_l} - \cdots - \Gamma^d{}_{ab_l}\,T^{a_1\ldots a_k}{}_{b_1\ldots b_{l-1}d}\,.\end{aligned} \tag{4.20}$$

Sometimes an alternative notation is used; just as commas are used to denote partial derivatives, semi-colons are used for the covariant derivatives:

$$\nabla_c T^{a_1\ldots a_k}{}_{b_1\ldots b_l} = T^{a_1\ldots a_k}{}_{b_1\ldots b_l;c}\,.$$

In these lectures we will mostly use ∇_a for the covariant derivative.

Still, we have not fully specified the connection and in fact, one can define many different connections on a manifold satisfying the previous four requirements. It turns out though that every metric defines a unique connection, which is the one that is used in general relativity.

To see this, the first thing to notice is that the difference between two connections, say Γ and $\tilde{\Gamma}$, is a tensor. Indeed,

$$\begin{aligned}\nabla_a V^b - \tilde{\nabla}_a V^b &= \partial_a V^b + \Gamma^b{}_{ac}\,V^c - \left(\partial_a V^b + \tilde{\Gamma}^b{}_{ac}\,V^c\right)\\ &= (\Gamma^b{}_{ac} - \tilde{\Gamma}^b{}_{ac})V^c\,.\end{aligned}$$

Since the left hand side is a tensor by definition of covariant derivative, the right hand side must also be a tensor. Hence,

$$\Gamma^b{}_{ac} - \tilde{\Gamma}^b{}_{ac} = S^b{}_{ac}$$

where $S^b{}_{ac}$ is a tensor. Next, notice that given a connection $\Gamma^c{}_{ab}$, one can form another connection by simply permuting the lower indices; namely the coefficients $\Gamma^c{}_{ba}$ also transform as (4.18) and hence they determine a distinct connection. Therefore, to every connection we can associate a tensor, known as the torsion tensor, given by

$$T^c{}_{ab} = \Gamma^c{}_{ab} - \Gamma^c{}_{ba} = 2\,\Gamma^c{}_{[ab]}\,. \tag{4.21}$$

It is clear that the torsion tensor in anti-symmetric in its lower indices, and a connection that is symmetric in its lower indices (and hence the torsion tensor vanishes) is called "torsion-free".

We can now determine the unique connection on a manifold with a metric g_{ab} by introducing two additional properties:

5. Torsion-free: $\Gamma^c{}_{ab} = \Gamma^c{}_{ba}$.

6. Metric compatibility: $\nabla_c g_{ab} = 0\,.$

The torsion free condition implies that covariant derivatives acting on a scalar field commute:

$$\nabla_a \nabla_b \phi = \nabla_b \nabla_a \phi\,.$$

A connection is metric compatible if the covariant derivative of the metric with respect to that connection vanishes everywhere, and such a connection is known as the Levi-Civita connection.

The metric-compatibility condition implies the following for the covariant derivative of the inverse metric:

$$\begin{aligned} 0 &= \nabla_a \delta^b_c \\ &= \nabla_a (g^{bd}\, g_{dc}) \\ &= g_{dc} \nabla_a g^{bd} + g^{bd} \nabla_a g_{dc} \\ &= \nabla_a (g^{bd})\, g_{dc} \\ \Rightarrow \nabla_a g^{bd} &= 0\,. \end{aligned}$$

In addition, a metric-compatible covariant derivative commutes with raising and lowering of indices. For example, for an arbitrary vector field V^a,

$$g_{ac} \nabla_b V^c = \nabla_b (g_{ac} V^c) = \nabla_b V_a\,.$$

Now we will show both the existence and uniqueness of the metric-compatible connection by explicitly determining the connection coefficients in terms of the metric. To do so, consider the following three expressions for the expanded metric compatibility condition obtained by permuting the indices:

$$\begin{aligned} \nabla_c g_{ab} &= \partial_c g_{ab} - \Gamma^d{}_{ca}\, g_{db} - \Gamma^d{}_{cb}\, g_{ad} = 0 \\ \nabla_a g_{bc} &= \partial_a g_{bc} - \Gamma^d{}_{ab}\, g_{dc} - \Gamma^d{}_{ac}\, g_{bd} = 0 \\ \nabla_b g_{ca} &= \partial_b g_{ab} - \Gamma^d{}_{bc}\, g_{da} - \Gamma^d{}_{ba}\, g_{cd} = 0 \end{aligned}$$

Subtracting the second and third equations from the first and using the symmetry properties of the connection, we obtain

$$\partial_c g_{ab} - \partial_a g_{bc} - \partial_b g_{ca} + 2\,\Gamma^d{}_{ab}\, g_{dc} = 0\,.$$

Multiplying this expression by g^{ec} and re-labelling the indices, we find the final expression for the Levi-Civita connection:

$$\Gamma^c{}_{ab} = \frac{1}{2}\, g^{cd} \left(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab} \right)\,. \tag{4.22}$$

This connection is also known as the Christoffel connection and the symbols in (4.22) are referred to as the Christoffel symbols.

Example: Christoffel symbols of the flat Euclidean metric in 2 dimensions in polar coordinates:

$$ds^2 = dr^2 + r^2\, d\theta^2\,.$$

The non-zero components of the metric are $g_{rr} = 1$, $g_{\theta\theta} = r^2$, while the non-zero components of the inverse metric are $g^{rr} = 1$ and $g^{\theta\theta} = \frac{1}{r^2}$. Notice that we use r and θ as indices in a notation that should be obvious; we will continue to do so in the rest of these lectures. Using (4.22), we compute

$$\begin{aligned}
\Gamma^r{}_{rr} &= \frac{1}{2}\, g^{ra}(\partial_r g_{ra} + \partial_r g_{ra} - \partial_a g_{rr}) \\
&= \frac{1}{2}\, g^{rr}(\partial_r g_{rr} + \partial_r g_{rr} - \partial_r g_{rr}) + \frac{1}{2}\, g^{r\theta}(\partial_r g_{r\theta} + \partial_r g_{r\theta} - \partial_\theta g_{rr}) \\
&= \frac{1}{2}(1)(0+0-0) + \frac{1}{2}(0)(0+0-0) \\
&= 0\,.
\end{aligned}$$

Similarly, we compute

$$\begin{aligned}
\Gamma^r{}_{\theta\theta} &= \frac{1}{2}\, g^{ra}(\partial_\theta g_{\theta a} + \partial_\theta g_{\theta a} - \partial_a g_\theta) \\
&= \frac{1}{2}\, g^{rr}(\partial_\theta g_{\theta r} + \partial_\theta g_{\theta r} - \partial_r g_{\theta\theta}) \\
&= -r\,.
\end{aligned}$$

Proceeding in exactly the same manner as above, we find that the remaining components of the Christoffel symbols are given by

$$\begin{aligned}
&\Gamma^r{}_{\theta r} = \Gamma^r{}_{r\theta} = 0 \\
&\Gamma^\theta{}_{rr} = 0 \\
&\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{1}{r} \\
&\Gamma^\theta{}_{\theta\theta} = 0\,.
\end{aligned}$$

Remark. The Christoffel symbols of the flat metric in Cartesian coordinates vanish identically (**exercise**).

4.6 Parallel transport and geodesics

In flat space, parallel transport of a vector along a curve intuitively means "keeping the vector constant" as we move it along the curve. More precisely, given a curve $x^b(\lambda)$, imposing that an arbitrary vector V^a is constant along this curve in flat space corresponds to:

$$\frac{d}{d\lambda} V^a = \frac{dx^b}{d\lambda}\frac{\partial}{\partial x^b} V^a = 0\,.$$

The crucial difference between flat and curved spaces is that, in a curved space, the result of parallel transporting a vector from one point to another will depend on the path taken between the points, see Fig. 4.4.

The generalisation of this concept to curved manifolds amounts to replace the partial derivative by a covariant derivative. Therefore, a vector V^a is said to be parallely transported along W^b if

$$W^b \nabla_b V^a = 0.$$

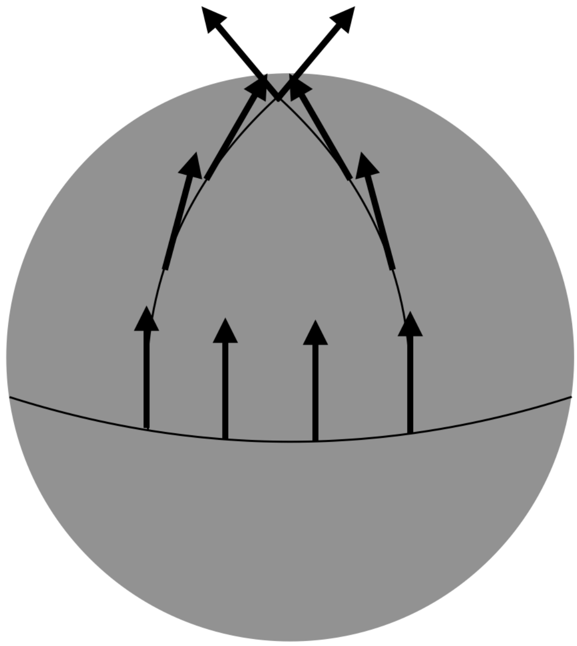


Figure 4.4: Parallel transport on the sphere. On a curved space, the result of parallel transporting a vector depends on the path taken.

This concept can be generalised to tensors of any rank. For example, a (k, l) tensor T is said to be parallel transported along a vector W^b if

$$W^b \nabla_b T^{a_1 a_2 \dots a_k}{}_{b_1 b_2 \dots b_l} = 0\,.$$

Now, recall that one way of characterising straight lines in Euclidean space is as curves whose tangent vectors are parallely transported at every point —i.e. they are autoparallels. The notion defined above can be used to define the analogue of *straight lines* in more general manifolds. Such curves are referred to as *affine geodesics* —i.e. curves along which the tangent vector is propagated parallely to itself. We remark that this definition can also be used to define a notion of shortest distance on a manifold using the metric, but we shall defer this to a later point in the course.

Letting W^b to be tangent to a geodesic, one has that

$$W^b \nabla_b W^a = 0,$$

from where

$$W^b \partial_b W^a + \Gamma^a{}_{cb} W^c W^b = 0.$$

If the curve is parametrised by λ, then

$$W^b = \frac{\mathrm{d}x^b}{\mathrm{d}\lambda},$$

and since

$$W^b \frac{\partial}{\partial x^b} = \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\equiv \frac{\mathrm{d}x^b}{\mathrm{d}\lambda} \frac{\partial}{\partial x^b} \right),$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\frac{\mathrm{d}x^a}{\mathrm{d}\lambda} \right) + \Gamma^a{}_{bc} \frac{\mathrm{d}x^c}{\mathrm{d}\lambda} \frac{\mathrm{d}x^b}{\mathrm{d}\lambda} = 0,$$

and finally that

$$\frac{\mathrm{d}^2x^a}{\mathrm{d}\lambda^2} + \Gamma^a{}_{bc}\frac{\mathrm{d}x^c}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda} = 0. \tag{4.23}$$

This the famous geodesic equation.

Note. From the existence and uniqueness theorems for ordinary differential equations, it follows that corresponding to every direction at a point, there exists a unique geodesic passing through the point. The initial conditions are

$$\lambda = \lambda_0, \quad x_0^a = x^a(0), \quad W_0^a = \frac{\mathrm{d}x^a}{\mathrm{d}\lambda}(0).$$

Example. Show that changing the geodesic parameter λ to σ in such a way that $\sigma = \sigma(\lambda)$, the geodesic equation only keeps its form (4.23) in σ if $\sigma = a\lambda + b$.

To see this recall that

$$\frac{\mathrm{d}x^a}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^a}{\mathrm{d}\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}\lambda},$$

so that

$$\frac{\mathrm{d}^2x^a}{\mathrm{d}\lambda^2} = \frac{\mathrm{d}^2x^a}{\mathrm{d}\sigma^2}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\lambda}\right)^2 + \frac{\mathrm{d}x^a}{\mathrm{d}\sigma}\frac{\mathrm{d}^2\sigma}{\mathrm{d}\lambda^2}.$$

Substituting into equation (4.23) one gets

$$\left(\frac{\mathrm{d}^2x^a}{\mathrm{d}\sigma^2} + \Gamma^a{}_{bc}\frac{\mathrm{d}x^c}{\mathrm{d}\sigma}\frac{\mathrm{d}x^b}{\mathrm{d}\sigma}\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\lambda}\right)^2 + \frac{\mathrm{d}x^a}{\mathrm{d}\sigma}\frac{\mathrm{d}^2\sigma}{\mathrm{d}\lambda^2} = 0,$$

which only has the form of (4.23) if

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}\lambda^2} = 0.$$

That is, if

$$\sigma = a\lambda + b.$$

A parameter of this form is called an *affine parameter*.

4.6.1 Metric geodesics

In Euclidean geometry, straight lines are defined as the shortest distance between any two points. Here we give an analogue of this for a manifold with a metric.

In Lorentzian manifolds, straight lines are not those with shortest distances (intervals) between 2 points, but the longest. The generalisation of a straight line —a geodesic line— turns out to be the curve of extremal path (i.e., maximal or minimal). In order to find extrema, one needs some elements of *calculus of variations*. Let

$$L = L(x, \dot{x}, \lambda), \quad x = x(\lambda), \quad \dot{x} = \frac{\mathrm{d}x}{\mathrm{d}\lambda}.$$

That is, L is a function of functions of λ —L is called a functional. It is assumed that L is differentiable in x, $\dot{x}$, λ.

We are looking for the necessary conditions on the function x such that the integral

$$\int_{x_1}^{x_2} L(x, \dot{x}, \lambda)\mathrm{d}\lambda$$

is stationary (i.e., a maximum or a minimum) with respect to changes in the function x. The required condition is called the *Euler-Lagrange equation* and takes the form

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0. \tag{4.24}$$

This expression can be generalised to the case where L is a function of N independent functions, $x^i(\lambda)$, $i = 1, \ldots, N$, provided they can be varied independently. In that case (4.24) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0, \tag{4.25}$$

corresponding to N equations, one for each value of i.

To deduce the geodesic equation we want to consider the length of the curve defined by $\int \mathrm{d}s$ to be stationary. Introducing a parameter λ along the curve such that

$$\int \mathrm{d}s = \int \frac{\mathrm{d}s}{\mathrm{d}\lambda}\mathrm{d}\lambda,$$

the problem becomes that of finding the extremals of

$$L = \frac{\mathrm{d}s}{\mathrm{d}\lambda} = \sqrt{g_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\frac{\mathrm{d}x^k}{\mathrm{d}\lambda}} = \sqrt{g_{jk}\dot{x}^i\dot{x}^j}.$$

Alternatively, one can find extremals of

$$L = \left(\frac{\mathrm{d}s}{\mathrm{d}\lambda}\right)^2 = g_{jk}\dot{x}^i\dot{x}^j.$$

A computation renders

$$\begin{aligned}
\frac{\partial L}{\partial \dot{x}^c} &= g_{ab}\frac{\partial \dot{x}^a}{\partial \dot{x}^c}\dot{x}^b + g_{ab}\dot{x}^a\frac{\partial \dot{x}^b}{\partial \dot{x}^c}, \\
&= g_{ab}\delta^a{}_c\dot{x}^b + g_{ab}\dot{x}^a\delta^b{}_c, \\
&= g_{cb}\dot{x}^b + g_{ac}\dot{x}^a = 2g_{ac}\dot{x}^a.
\end{aligned}$$

Now, recall that the chain rule gives

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^e}{\mathrm{d}\lambda}\frac{\partial}{\partial x^e} = \dot{x}^e\partial_e.$$

Thus,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial \dot{x}^c}\right) &= 2\frac{\mathrm{d}g_{ac}}{\mathrm{d}\lambda}\dot{x}^a + 2g_{ac}\frac{\mathrm{d}\dot{x}^a}{\mathrm{d}\lambda} \\
&= 2\partial_e g_{ac}\dot{x}^e\dot{x}^a + 2g_{ac}\ddot{x}^a.
\end{aligned}$$

Finally,

$$\frac{\partial L}{\partial x^c} = \partial_c g_{ab}\dot{x}^a\dot{x}^b.$$

Thus, one has that

$$0 = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial \dot{x}^a}\right) - \frac{\partial L}{\partial x^a} = 2g_{ac}\ddot{x}^a + (\partial_b g_{ac} + \partial_a g_{bc} - \partial_c g_{ab})\dot{x}^a\dot{x}^b.$$

Multiplying by $\frac{1}{2}g^{fc}$ one obtains

$$\ddot{x}^f + \Gamma^f{}_{ab}\dot{x}^a\dot{x}^b = 0,$$

which can be rewritten as

$$\frac{\mathrm{d}^2 x^f}{\mathrm{d}\lambda^2} + \Gamma^f_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda} = 0. \tag{4.26}$$

This is the geodesic equation which we have met already. Thus, "straight lines" are also extremal.

Remark 1. In Euclidean space in Cartesian coordinates or in Minkowski space in Minkowski coordinates all the Christoffel symbols vanishes and equation (4.26) becomes

$$\frac{\mathrm{d}^2 x^l}{\mathrm{d}s^2} = 0,$$

which is the usual equation for straight motion.

Remark 2. As it stands, the above equation only makes sense for spacelike curves for which $\mathrm{d}s^2 > 0$. For timelike curves one uses $\mathrm{d}\tau$ instead. Also, starting with $\int \mathrm{d}s^2$ gives the same geodesic equation.

Remark 3. For null geodesics, i.e. geodesics for which $\mathrm{d}s = 0$, the curve may be parametrised by a parameter

$$\frac{\mathrm{d}^2 x^l}{\mathrm{d}u^2} + \Gamma^l{}_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}u}\frac{\mathrm{d}x^k}{\mathrm{d}u} = 0,$$

where

$$g_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}u}\frac{\mathrm{d}x^k}{\mathrm{d}u} = 0.$$

Remark 4. It can be proved that if g_{ab} is Riemannian then the solutions to equation (4.26) are curves of minimum length. On the other hand, if g_{ab} is Lorentzian, then the geodesics maximise length. Now, recall that in Special Relativity one defines the proper time as $\mathrm{d}\tau^2 = -\mathrm{d}s^2/c^2$. Thus, time observed by a comoving clock always goes slower.

4.6.2 An example of the use of the Euler-Lagrange equations

The Euler-Lagrange equations can be used to compute in a more efficient way the Christoffel symbols. As a way of comparison, consider again the line element of the 2-dimensional sphere.

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2.$$

The Lagrangian is in this case given by

$$L = \left(\frac{\mathrm{d}s}{\mathrm{d}\lambda}\right)^2 = \left(\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\right)^2 + \sin^2\theta\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\lambda}\right)^2 = \dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2.$$

The Euler-Lagrange equations are given by

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0\,,$$

with $(x^1, x^2) = (\theta, \phi)$. Let's consider the different components of these equations. For $i = 1$ one has

$$\frac{\partial L}{\partial \theta} = 2\sin\theta\cos\theta\dot{\varphi}^2, \quad \frac{\partial L}{\partial \dot{\theta}} = 2\dot{\theta}, \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 2\ddot{\theta}$$

and finally that

$$\ddot{\theta} - \sin\theta\cos\theta\dot{\varphi}^2 = 0. \tag{4.27}$$

The latter is equivalent to (cfr. (4.26)):

$$\frac{\mathrm{d}^2 x^1}{\mathrm{d}s^2} + \Gamma^1{}_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}s}\frac{\mathrm{d}x^k}{\mathrm{d}s} = 0,$$

or

$$\frac{\mathrm{d}^2 x^1}{\mathrm{d}s^2} + \Gamma^1{}_{11}\frac{\mathrm{d}x^1}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s} + \Gamma^1{}_{12}\frac{\mathrm{d}x^1}{\mathrm{d}s}\frac{\mathrm{d}x^2}{\mathrm{d}s} + \Gamma^1{}_{21}\frac{\mathrm{d}x^2}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s} + \Gamma^1{}_{22}\frac{\mathrm{d}x^2}{\mathrm{d}s}\frac{\mathrm{d}x^2}{\mathrm{d}s} = 0.$$

However, in our case one only has $\dot{\phi}^2$ terms so the latter becomes

$$\frac{\mathrm{d}^2 x^1}{\mathrm{d}s^2} + \Gamma^1{}_{22}\left(\frac{\mathrm{d}x^2}{\mathrm{d}s}\right)^2 = 0.$$

The latter in combination with (4.27) gives

$$\Gamma^1{}_{22} = -\sin\theta\cos\theta, \quad \Gamma^1{}_{11} = \Gamma^1{}_{12} = \Gamma^1{}_{21} = 0.$$

For $i = 2$ one finds from

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(2\dot{\varphi}\sin^2\theta) = 0,$$

that

$$\ddot{\varphi} + 2\cot\theta\dot{\theta}\dot{\varphi} = 0. \tag{4.28}$$

Again, from the equation for the geodesic one has that

$$\frac{\mathrm{d}^2 x^2}{\mathrm{d}s^2} + \Gamma^2{}_{12}\frac{\mathrm{d}x^1}{\mathrm{d}s}\frac{\mathrm{d}x^2}{\mathrm{d}s} + \Gamma^2{}_{21}\frac{\mathrm{d}x^2}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s} = 0.$$

However,

$$\Gamma^2{}_{12} = \Gamma^2{}_{21},$$

and hence

$$\Gamma^2{}_{12} = \Gamma^2{}_{21} = \cot\theta.$$

Finally,

$$\Gamma^2{}_{22} = \Gamma^2{}_{11} = 0.$$

The solution to the geodesic equations

As a final remark it is noticed that a solution to the geodesic equations found in the previous section is given by

$$\theta = \lambda, \quad \varphi = \text{constant}.$$

his corresponds to a geodesic starting at the North Pole. Notice that the geodesic moves towards the Equator along a meridian. This shows that geodesics on the sphere move along the great circles of the sphere —the circles obtained by intersecting the sphere by a plane that passes through the centre of the sphere.

Chapter 5

Curvature

A novel feature of General Relativity is that it employs the notion of *curved space*. Our intuition of curvature is mainly based on the curvature of 2-dimensional objects in 3-dimensional space, like spheres, saddles, etc. The notion of curvature whose definition depends on a space of higher dimension is called *extrinsic*. In the case of spacetime this notion is not useful and require an *intrinsic* notion —i.e. a definition which is independent of the embedding space.

5.1 The Riemann curvature tensor

Having studied covariant derivatives and parallel transport, we are now ready to discuss curvature. The curvature is quantified by the Riemann tensor, which is computed from the connection. The idea behind this is that we know what we mean by a "flat" connection – the usual Christoffel connection for the Euclidean or Minkowski metric in Cartesian coordinates. In addition, in flat space we know that parallel transport around a closed loop leaves a vector unchanged, the covariant derivatives of tensors commute and initially parallel geodesics remain parallel. The Riemann tensor appears when we study those aspects of the connection in curved spaces.

We have already seen in the case of the 2-sphere that the parallel transport of a vector around a closed loop leads to a transformation of the vector. The resulting transformation depends on the total curvature enclosed by the loop. However, it is more useful to have a local notion of curvature. To do so, it is conventional to consider the parallel transport of a vector along an infinitesimal loop. Consider two infinitesimal vectors A^a and B^b and suppose that we parallel transport an arbitrary vector V^a first in the direction of A^a, then along B^b, then backwards along A^a and B^b to return to the starting point, see Fig. 5.1 (left).The change in the vector V^a, denoted by δV^a, is given by

$$\delta V^c = R^c{}_{dab}\, V^d\, A^a\, B^b\,. \tag{5.1}$$

where $R^c{}_{dab}$ is the Riemann tensor. This tensor is antisymmetric in the last two indices:

$$R^c{}_{dab} = -R^c{}_{dba}\,.$$

This has to be the case because interchanging the vectors A^a and B^b corresponds to traversing the loop in the opposite direction and it hence it should give the inverse of the original answer. If (5.1) is taken as the definition of the Riemann tensor, it implies certain choice of convention for the ordering of the indices.

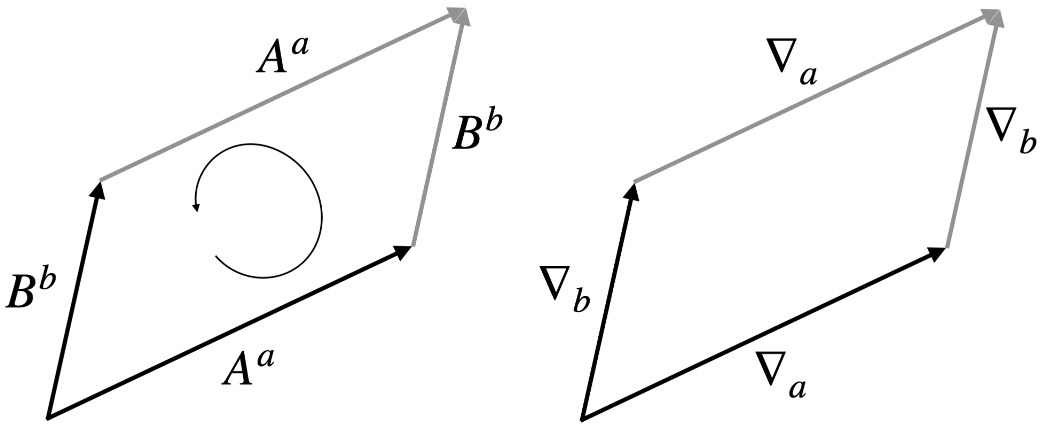


Figure 5.1: *Left:* Infinitesimal loop defined by the vectors A^a and B^b. *Right:* commutator of two covariant derivatives.

A related (and equivalent) way of defining the Riemann tensor is to consider the commutator of two covariant derivatives. The commutator of two covariant derivatives measures the difference between parallel transporting a given tensor first in one direction and then the other, versus the opposite ordering, see Fig. 5.1 (right). The computation goes as follows; consider an arbitrary vector field V^a, then

$$\begin{aligned}
[\nabla_a,\,\nabla_b]V^c &= \nabla_a\nabla_b V^c - \nabla_b\nabla_a V^c \\
&= \partial_a(\nabla_b V^c) - \Gamma^d{}_{ab}\nabla_d V^c + \Gamma^c{}_{ad}\nabla_b V^d - (a\leftrightarrow b) \\
&= \partial_a\partial_b V^c + (\partial_a\Gamma^c{}_{bd})V^d + \Gamma^c{}_{bd}\partial_a V^d - \Gamma^d{}_{ab}\partial_d V^c - \Gamma^d{}_{ab}\Gamma^c{}_{de}V^e \\
&\quad + \Gamma^c{}_{ad}\partial_b V^d + \Gamma^c{}_{ad}\Gamma^d{}_{be}V^e - (a\leftrightarrow b) \\
&= (\partial_a\Gamma^c{}_{bd} - \partial_b\Gamma^c_{ad} + \Gamma^c{}_{ae}\Gamma^e{}_{bd} - \Gamma^c{}_{be}\Gamma^e_{ad})V^d\,.
\end{aligned} \tag{5.2}$$

In the last step we have relabelled some dummy indices and eliminated terms that cancel by antisymmetry. Since the left hand side of this expression is a tensor, so must be the right hand side. We write,

$$[\nabla_a,\,\nabla_b]V^c = R^c{}_{dab}\,V^d\,, \tag{5.3}$$

where the Riemann tensor is defined as

$$R^c{}_{dab} = \partial_a\Gamma^c{}_{bd} - \partial_b\Gamma^c_{ad} + \Gamma^c{}_{ae}\Gamma^e{}_{bd} - \Gamma^c{}_{be}\Gamma^e_{ad}\,. \tag{5.4}$$

Notice that the Riemann tensor (5.4) is constructed from non-tensorial quantities (i.e., the Chrisoffel symbols and their derivatives) but it transforms as a tensor under general coordinate transformations (**check!**).

Using the fact that $[\nabla_a,\,\nabla_b](W_cV^c) = 0$ because W_cV^c is a scalar, we find

$$[\nabla_a,\,\nabla_b]W_c = -R^d{}_{cab}\,W_d\,. \tag{5.5}$$

Proceeding by induction, we can compute the action of $[\nabla_c,\,\nabla_d]$ on a tensor of arbitrary rank. One finds,

$$\begin{aligned}
[\nabla_c,\,\nabla_d]X^{a_1\ldots a_k}{}_{b_1\ldots b_l} &= R^{a_1}{}_{ecd}X^{ea_2\ldots a_k}{}_{b_1\ldots b_l} + \ldots + R^{a_k}{}_{ecd}X^{a_1\ldots a_{k-1}e}{}_{b_1\ldots b_l} \\
&\quad - R^e{}_{b_1cd}X^{a_1\ldots a_k}{}_{eb_2\ldots b_l} - \ldots - R^e{}_{b_lcd}X^{a_1\ldots a_k}{}_{b_1\ldots b_{l-1}e}
\end{aligned} \tag{5.6}$$

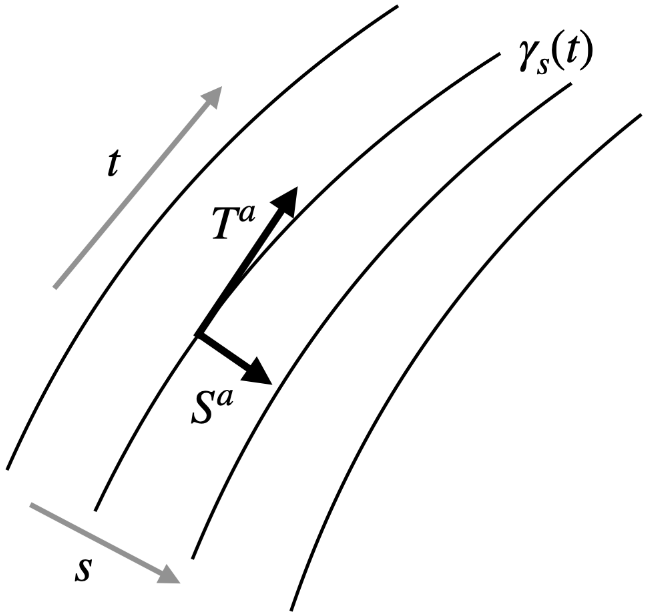


Figure 5.2: A set of geodesics $\gamma_s(t)$ with tangent vectors T^a. The vector field S^a measures the deviation between nearby geodesics.

5.2 Geodesic deviation

The defining property of Euclidean (flat) geometry is the parallel postulate: initially parallel lines remain parallel forever. In a curved space this is not true; for instance, on a sphere we have seen that initially parallel geodesics will eventually cross. Quantifying this behaviour on an arbitrary curved space is not straightforward because the notion of "parallel" does not extend naturally from flat to curved spaces. The best one can do is to consider geodesic curves that are initially parallel and see how they behave as we travel along the geodesics.

Consider a one parameter family of geodesics $\gamma_s(t)$ so that for each $s \in \mathbb{R}$, γ_s is a geodesic parametrised by the affine parameter t. The collection of such curves defines a smooth two-dimensional surface; the parameters (s,t) may be chosen as coordinates on this surface provided that the geodesics do not cross. The entire surface is the set of points $x^a(s,t) \in M$. There are two natural vector fields on this surface; the tangent vectors to the geodesics,

$$T^a = \frac{\partial x^a}{\partial t}\,,$$

and the deviation vectors

$$S^a = \frac{\partial x^a}{\partial s}\,.$$

This name comes from the (informal) notion that S^a points from one geodesic towards the neighbouring ones. See Fig. 5.2. This idea leads to define the "relative velocity of geodesics",

$$V^a = (\nabla_T S)^a = T^b \nabla_b S^a\,,$$

and the "relative acceleration of geodesics",

$$A^a = (\nabla_T V)^a = T^b \nabla_b V^a\,.$$

Since S and T are basis vectors adapted to a coordinate system, their commutator vanishes,

$$[S, T] = 0 \Rightarrow S^b \nabla_b T^a = T^b \nabla_b S^a \,.$$

Keeping this in mind, we can explicitly compute the relative acceleration of geodesics:

$$\begin{aligned}
A^a &= T^b \nabla_b V^a = T^b \nabla_b (T^c \nabla_c S^a) \\
&= T^b \nabla_b (S^c \nabla_c T^a) \\
&= (T^b \nabla_b S^c)(\nabla_c T^a) + T^b\, S^c \nabla_b \nabla_c T^a \\
&= (T^b \nabla_b S^c)(\nabla_c T^a) + T^b\, S^c (\nabla_c \nabla_b T^a + R^a{}_{dbc}\, T^d) \\
&= (T^b \nabla_b S^c)(\nabla_c T^a) + S^c \nabla_c (T^b \nabla_b T^a) - (S^c \nabla_c T^b) \nabla_b T^a + R^a{}_{dbc}\, T^d\, T^b\, S^c \\
&= R^a{}_{dbc}\, T^d\, T^b\, S^c \,.
\end{aligned} \tag{5.7}$$

The first line is just the definition of A^a and the second line comes from $[S, T] = 0$. The third line is just the Leibniz rule; the fourth line replaces a double covariant derivative by derivatives in the opposite order plus the Riemann tensor. The fifth line uses again the Leibniz rule (in the opposite order than usual), and then we cancel two identical terms and notice that the term $T^b \nabla_b T^a$ vanishes because T^a is the tangent vector to a geodesic. The result,

$$A^a = \nabla_T \nabla_T S^a = R^a{}_{dbc}\, T^d\, T^b\, S^c \,, \tag{5.8}$$

is the geodesic deviation equation. It expresses that the relative acceleration between two neighbouring geodesics is proportional to the curvature. Physically the acceleration of neighbouring geodesics is interpreted as a manifestation of the gravitational tidal forces.

5.3 Symmetries of the curvature tensor

In general, a tensor of rank 4 has $4^4 = 256$ components (in spacetime). Symmetries, if present, are important because they reduce the number of independent components. Lowering the index in the definition of the Riemann tensor one obtains

$$R_{abcd} = g_{af}(\partial_c \Gamma^f{}_{bd} - \partial_d \Gamma^f{}_{bc}) + \Gamma_{aec}\Gamma^e{}_{bd} - \Gamma_{aed}\Gamma^e{}_{bc},$$

where

$$R_{abcd} = g_{af} R^f{}_{bcd}, \quad \Gamma_{abd} = g_{af}\Gamma^f{}_{bd}.$$

Now, since R_{abcd} is a tensor, it should have the same symmetries in all frames. Accordingly, choose a *locally inertial frame* for which the Christoffel symbols vanish. For these coordinates one has then that

$$R_{\hat{a}\hat{b}\hat{c}\hat{d}} = g_{\hat{a}\hat{f}}(\partial_{\hat{c}} \Gamma^{\hat{f}}{}_{\hat{b}\hat{d}} - \partial_{\hat{d}} \Gamma^{\hat{f}}{}_{\hat{b}\hat{c}}).$$

where we use hatted indices $\hat{a}, \ldots$ to denote that these expressions are only valid in locally inertial coordinates. Recalling that

$$\Gamma_{abc} = \tfrac{1}{2}\left(\partial_b g_{ca} + \partial_c g_{ba} - \partial_a g_{bc}\right)$$

one obtains

$$R_{\hat{a}\hat{b}\hat{c}\hat{d}} = \tfrac{1}{2}\left(\partial_{\hat{b}}\partial_{\hat{c}} g_{\hat{a}\hat{d}} + \partial_{\hat{a}}\partial_{\hat{d}} g_{\hat{b}\hat{c}} - \partial_{\hat{a}}\partial_{\hat{c}} g_{\hat{b}\hat{d}} - \partial_{\hat{b}}\partial_{\hat{d}} g_{\hat{a}\hat{c}}\right),$$

from where it is easy to read the symmetries of the tensor. It can be checked that

$$R_{abcd} = -R_{bacd}, \quad R_{abcd} = -R_{abdc}, \quad R_{abcd} = R_{cdab}.$$

Furthermore,

$$R_{abcd} + R_{adbc} + R_{acdb} = 0 \quad \Rightarrow \quad R_{a[bcd]} = 0.$$

These symmetries amount to 236 constraints, so R_{abcd} has only 20 non-zero components.

5.4 Bianchi identities, the Ricci and Einstein tensors

Recall that in a locally inertial frame one had that

$$R_{\hat{c}\hat{d}\hat{a}\hat{b}} = \tfrac{1}{2}\left(\partial_{\hat{a}}\partial_{\hat{d}}g_{\hat{c}\hat{b}} - \partial_{\hat{a}}\partial_{\hat{c}}g_{\hat{b}\hat{d}} - \partial_{\hat{b}}\partial_{\hat{d}}g_{\hat{c}\hat{a}} + \partial_{\hat{b}}\partial_{\hat{c}}g_{\hat{a}\hat{d}}\right) .$$

Differentiating with respect to $\hat{x}^e$ one obtains

$$\partial_{\hat{e}}R_{\hat{c}\hat{d}\hat{a}\hat{b}} = \tfrac{1}{2}\partial_{\hat{e}}\left(\partial_{\hat{a}}\partial_{\hat{d}}g_{\hat{c}\hat{b}} - \partial_{\hat{a}}\partial_{\hat{c}}g_{\hat{b}\hat{d}} - \partial_{\hat{b}}\partial_{\hat{d}}g_{\hat{c}\hat{a}} + \partial_{\hat{b}}\partial_{\hat{c}}g_{\hat{a}\hat{d}}\right) .$$

Now consider the sum of the cyclic permutations of the first three indices:

$$\begin{aligned}
&\partial_{\hat{e}}R_{\hat{c}\hat{d}\hat{a}\hat{b}} + \partial_{\hat{c}}R_{\hat{d}\hat{e}\hat{a}\hat{b}} + \partial_{\hat{d}}R_{\hat{e}\hat{c}\hat{a}\hat{b}} \\
&\quad = \tfrac{1}{2}\big(\partial_{\hat{e}}\partial_{\hat{a}}\partial_{\hat{d}}g_{\hat{c}\hat{b}} - \partial_{\hat{e}}\partial_{\hat{a}}\partial_{\hat{c}}g_{\hat{b}\hat{d}} - \partial_{\hat{e}}\partial_{\hat{b}}\partial_{\hat{d}}g_{\hat{c}\hat{a}} + \partial_{\hat{e}}\partial_{\hat{b}}\partial_{\hat{c}}g_{\hat{a}\hat{d}} \\
&\qquad + \partial_{\hat{c}}\partial_{\hat{a}}\partial_{\hat{e}}g_{\hat{d}\hat{b}} - \partial_{\hat{c}}\partial_{\hat{a}}\partial_{\hat{d}}g_{\hat{b}\hat{e}} - \partial_{\hat{c}}\partial_{\hat{b}}\partial_{\hat{e}}g_{\hat{d}\hat{a}} + \partial_{\hat{c}}\partial_{\hat{b}}\partial_{\hat{d}}g_{\hat{a}\hat{e}} \\
&\qquad + \partial_{\hat{d}}\partial_{\hat{a}}\partial_{\hat{c}}g_{\hat{e}\hat{b}} - \partial_{\hat{d}}\partial_{\hat{a}}\partial_{\hat{e}}g_{\hat{b}\hat{c}} - \partial_{\hat{d}}\partial_{\hat{b}}\partial_{\hat{c}}g_{\hat{e}\hat{a}} + \partial_{\hat{d}}\partial_{\hat{b}}\partial_{\hat{e}}g_{\hat{a}\hat{c}}\big) \\
&\quad = 0\,.
\end{aligned} \tag{5.9}$$

Since this is an equation between tensors, it is true in any coordinate system, even though we derived it in a particular one. By the antisymmetry property $R_{cdab} = -R_{dcab}$, we can re-write this equation as

$$\nabla_e R_{cdab} + \nabla_d R_{ecab} + \nabla_c R_{deab} = 0 \quad \Rightarrow \quad \nabla_{[e}R_{cd]ab} = 0\,. \tag{5.10}$$

This tensorial equation is valid in all frames and is called the *Bianchi identity.* One could have derived it by directly taking the covariant derivative of the Riemann tensor.

The Ricci tensor

The Ricci tensor is obtained by contracting the first and third indices of the Riemann tensor:

$$\begin{aligned}
R_{ab} &\equiv g^{cd}R_{cadb} = R^c{}_{acb} \\
&= \partial_c\Gamma^c{}_{ab} - \partial_a(\Gamma^c{}_{cb}) + \Gamma^d{}_{ab}\Gamma^c{}_{cd} - \Gamma^d{}_{ca}\Gamma^c{}_{db}\,.
\end{aligned} \tag{5.11}$$

Remark 1. Because of the symmetries of the Riemann tensor one has that the Ricci tensor is symmetric. That is,

$$R_{ab} = R_{ba}\,.$$

Remark 2. Other contractions of the Riemann tensor vanish or give $\pm R_{ab}$. For example $R^c{}_{cab} = 0$ since R_{cdab} is anti-symmetric in c and d. Also,

$$R^c{}_{abc} = -R^c{}_{acb} = -R_{ab}\,,$$

and so on.

Remark 3. One can show that

$$\Gamma^a{}_{ab} = \partial_b \ln\sqrt{|g|}\,,$$

where $g = \det(g_{ab})$. Therefore, we have the following formula for the Ricci tensor:

$$R_{ab} = \partial_c\Gamma^c{}_{ab} - \partial_a\partial_b\ln\sqrt{|g|} + \Gamma^c{}_{ab}\partial_c\ln\sqrt{|g|} - \Gamma^d{}_{ca}\Gamma^c{}_{db}\,. \tag{5.12}$$

The Ricci scalar

The Ricci scalar is defined as the contraction of the indices of the Ricci tensor:

$$R \equiv g^{ab} R_{ab} = g^{ac} g^{bd} R_{abcd}.$$

The Einstein tensor

In the next computations recall that $\nabla_c g_{ab} = 0$ and $\nabla_c g^{ab} = 0$ since the Christoffel connection is metric compatible. Contract twice the Bianchi identity (5.10),

$$\begin{aligned} 0 &= g^{bd} g^{ae} (\nabla_e R_{cdab} + \nabla_c R_{deab} + \nabla_d R_{ecab}) \\ &= \nabla^a R_{ca} - \nabla_c R + \nabla^b R_{cb} \,, \end{aligned} \tag{5.13}$$

or

$$\nabla^a R_{ac} = \frac{1}{2} \nabla_c R \,. \tag{5.14}$$

Note that, unlike the partial derivative, it makes sense to raise an index on the covariant derivative of a tensor because it is another tensor and due to the metric compatibility. We define the *Einstein tensor* as

$$G_{ab} \equiv R_{ab} - \frac{1}{2} R \, g_{ab} \,, \tag{5.15}$$

We then see that the twice-contracted Bianchi identity (5.14) is equivalent to

$$\nabla^a G_{ab} = 0 \,. \tag{5.16}$$

Remark 1. The Einstein tensor, which is symmetric due to the symmetry of the Ricci tensor and the metric, has 10 independent components and it will play a crucial role in general relativity.

Remark 2. By construction, the Einstein tensor is *divergence free*.

The Weyl tensor

The Ricci tensor and the Ricci scalar contain all the information about the possible contractions of the Riemann tensor. The remaining information, namely the trace-free parts, are captured by the Weyl tensor. This tensor is defined as the Riemann tensor with all the contractions removed. In an n-dimensional manifold, one has

$$C_{abcd} = R_{abcd} - \frac{2}{n-2} (g_{a[c} R_{d]b} - g_{b[c} R_{d]a}) + \frac{2}{(n-1)(n-2)} \, g_{a[c} g_{d]b} \, R \,,$$

and hence

$$C^a{}_{bac} = 0 \,.$$

Remark 1. By construction, the Weyl tensor has the same symmetries as the Riemann tensor:

$$C_{abcd} = C_{[ab][cd]} \,, \quad C_{abcd} = C_{cdab} \,, \quad C_{a[bcd]} = 0 \,.$$

Remark 2. The Weyl tensor is only defined in three or more dimensions; in three dimensions it vanishes identically.

Remark 3. A very important property of the Weyl tensor is that it is invariant under conformal transformations of the metric, $g_{ab} \to \Omega(x)^2 \, g_{ab}$, where $\Omega(x)$ is an arbitrary function of the spacetime coordinates.

Example: curvature tensors of the 2-sphere. Consider a round 2-sphere of radius a with metric

$$ds^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,.$$

The non-zero Christoffel symbols are given by

$$\begin{aligned} \Gamma^{\theta}{}_{\phi\phi} &= -\sin\theta\,\cos\theta\,, \\ \Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} &= \cot\theta\,. \end{aligned}$$

Given the symmetries of the Riemann tensor, the only non-trivial component (up to symmetries) is:

$$\begin{aligned} R^{\theta}{}_{\phi\theta\phi} &= \partial_\theta\Gamma^{\theta}{}_{\phi\phi} - \partial_\phi\Gamma^{\theta}{}_{\theta\phi} + \Gamma^{\theta}{}_{\theta b}\,\Gamma^{b}{}_{\phi\phi} - \Gamma^{\theta}{}_{\phi b}\,\Gamma^{b}{}_{\theta\phi} \\ &= (\sin^2\theta - \cos^2\theta) - (0) + (0) - (-\sin\theta\,\cos\theta)(\cot\theta) \\ &= \sin^2\theta\,. \end{aligned}$$

Lowering the first index gives

$$\begin{aligned} R_{\theta\phi\theta\phi} &= g_{\theta c}\,R^{c}{}_{\phi\theta\phi} \\ &= g_{\theta\theta}\,R^{\theta}{}_{\phi\theta\phi} \\ &= a^2\,\sin^2\theta\,. \end{aligned}$$

The Ricci tensor is then computed from $R_{ab} = g^{cd}\,R_{cadb}$, which gives

$$\begin{aligned} R_{\theta\theta} = g^{\phi\phi}\,R_{\phi\theta\phi\theta} &= 1 \\ R_{\theta\phi} = R_{\phi\theta} &= 0 \\ R_{\phi\phi} = g^{\theta\theta}\,R_{\theta\phi\theta\phi} &= \sin^2\theta\,. \end{aligned}$$

Finally, the Ricci scalar is given by,

$$R = g^{ab}\,R_{ab} = g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} = \frac{2}{a^2}\,.$$

Note that the scalar of curvature, i.e., the Ricci scalar, decreases as the radius of the sphere increases. In more general cases, we will sometimes refer to the "radius of curvature" of a manifold as providing a length scale over which the curvature varies; the larger the radius of curvature, the smaller the curvature is.

Chapter 6

General Relativity

6.1 Towards the Einstein equations

There are several ways of motivating the Einstein equations. The most natural is perhaps through considerations involving the Equivalence Principle. In gravitational fields there exist local inertial frames in which Special Relativity is recovered. The equation of motion of a free particle in such frames is:

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}\tau^2} = 0. \tag{6.1}$$

Relative to an arbitrary (accelerating frame) specified by $x'^a = x'^a(x^b)$, the latter becomes:

$$\frac{\mathrm{d}^2 x'^a}{\mathrm{d}\tau^2} + \gamma^a{}_{bc} \frac{\mathrm{d}x'^b}{\mathrm{d}\tau} \frac{\mathrm{d}x'^c}{\mathrm{d}\tau} = 0,$$

where

$$\gamma^a{}_{bc} = \frac{\partial x'^a}{\partial x^d} \frac{\partial^2 x^d}{\partial x'^b \partial x'^c}.$$

Here the $\gamma^a{}_{bc}$ are the "fictitious" terms that arise due to the non-inertial nature of the frame. Now, due to the Equivalence Principle the latter implies that locally gravity is equivalent to acceleration and this in turn gives rise to non-inertial frames. The main idea of General relativity is to argue that gravitation as well as inertial forces should be described by appropriate $\gamma^a{}_{bc}$'s!

Clearly (6.1) is not a tensorial equation since it is not left invariant upon changing frame: although $\frac{dx^a}{d\tau}$ is a well-defined vector, $\frac{d^2x^a}{d\tau^2}$ is not. Note that we can use the chain rule $\frac{d}{d\tau} = \frac{dx^b}{d\tau}\frac{\partial}{\partial x^b}$ to write

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}\tau^2} = \frac{\mathrm{d}x^b}{\mathrm{d}\tau} \partial_b \left(\frac{\mathrm{d}x^a}{\mathrm{d}\tau} \right)$$

Now it is clear how we can generalise equation (6.1) to curved space: we simply replace the partial derivative by a covariant derivative:

$$\frac{\mathrm{d}x^b}{\mathrm{d}\tau} \partial_b \left(\frac{\mathrm{d}x^a}{\mathrm{d}\tau} \right) \to \frac{\mathrm{d}x^b}{\mathrm{d}\tau} \nabla_b \left(\frac{\mathrm{d}x^a}{\mathrm{d}\tau} \right) = \frac{\mathrm{d}^2 x^a}{\mathrm{d}\tau^2} + \Gamma^a{}_{bc} \frac{\mathrm{d}x^b}{\mathrm{d}\tau} \frac{\mathrm{d}x^c}{\mathrm{d}\tau}.$$

Therefore, we conclude that the generalisation of (6.1) to curved spaces is

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}\tau^2} + \Gamma^a{}_{bc} \frac{\mathrm{d}x^b}{\mathrm{d}\tau} \frac{\mathrm{d}x^c}{\mathrm{d}\tau} = 0. \tag{6.2}$$

To see that (6.2) indeed describes the motion of test particles in gravitational fields, we can consider the Newtonian limit of this equation. More precisely, in Newtonian limit we assume that particles are moving slowly compared to the speed of light, gravitational fields are weak (so it can be considered as a perturbation of flat space) and that the gravitational field is static. Taking the proper time τ as an affine parameter along the geodesic, "moving slowly" means

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau} \ll \frac{\mathrm{d}t}{\mathrm{d}\tau}\,.$$

where $i = 1, 2, 3$ denotes the spatial coordinates. In this limit, the geodesic equation (6.2) becomes

$$\frac{\mathrm{d}^2x^a}{\mathrm{d}\tau^2} + \Gamma^a{}_{00}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 = 0. \tag{6.3}$$

Since the gravitational field is assumed to be static, all t-derivatives of g_{ab} vanish ($\partial_0 g_{ab} = 0$) and the relevant Christoffel symbols simplify

$$\begin{aligned}\Gamma^a{}_{00} &= \tfrac{1}{2}g^{ab}(\partial_0 g_{b0} + \partial_0 g_{0b} - \partial_b g_{00})\\ &= -\tfrac{1}{2}g^{ab}\partial_b g_{00}\end{aligned} \tag{6.4}$$

Furthermore, since the field is weak, one may adopt a local coordinate system in which

$$g_{ab} = \eta_{ab} + h_{ab}, \quad |h_{ab}| \ll 1. \tag{6.5}$$

From the definition of the inverse metric, $g^{ab}g_{bc} = \delta^a_b$, we find that to first order in h_{ab},

$$g^{ab} = \eta^{ab} - h^{ab}$$

where $h^{ab} = \eta^{ac}\eta^{bd}h_{cd}$. Substituting this into (6.4) and expanding to first order in h_{ab}, one has that

$$\Gamma^a{}_{00} = -\tfrac{1}{2}\eta^{ad}\partial_d h_{00}.$$

Therefore, in this limit the geodesic equation (6.3) becomes:[1]

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}\tau^2} = \frac{1}{2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2\partial_i h_{00}, \tag{6.6a}$$

$$\frac{\mathrm{d}^2t}{\mathrm{d}\tau^2} = 0, \qquad \text{as} \qquad \partial_0 h_{00} = 0. \tag{6.6b}$$

From (6.6b) it follows that $\frac{dt}{d\tau}$ is a constant. Also, from

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau} = \frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau},$$

it follows that

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}\tau^2} = \frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}^2t}{\mathrm{d}\tau^2},$$

which in our case reduces to

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}\tau^2} = \frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2.$$

[1] Note that $\eta^{ij} = \delta^{ij}$ so spatial indices upstairs and downstairs are the same.

Combining the latter with (6.6a) we obtain

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = \frac{1}{2}\partial_i h_{00}. \tag{6.7}$$

The corresponding Newtonian result is

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\partial_i \phi \tag{6.8}$$

where ϕ is the gravitational potential. Far from a central body of mass M at a distance r, ϕ is given by

$$\phi = -\frac{GM}{r},$$

where G is Newton's constant of gravitation. Comparing (6.7) and (6.8) one finds that

$$h_{00} = -2\phi + \text{constant}.$$

However, at large distances from M one has that $\phi \to 0$ (gravity becomes negligible) and $g_{ab} \to \eta_{ab}$ (the space becomes flat). Therefore the constant must be zero and we can conclude that

$$h_{00} = -2\phi. \tag{6.9}$$

Substituting in (6.5) one finds

$$g_{00} = -(1 + 2\phi). \tag{6.10}$$

Now, recall that ϕ has dimensions of (velocity)^2, $[\phi] = [GM/R] = L^2/T^2$. Therefore one has that ϕ/c^2 at the surface of the Earth is $\sim 10^{-9}$, on the surface of the Sun is $\sim 10^{-6}$, at the surface of a white dwarf is $\sim 10^{-4}$ while at the surface of a neutron start is $\sim 10^{-2}$. On the other hand, at horizon of a black hole $\phi/c^2 \sim 1$. It follows that in most cases the distortion produced by gravity in the spacetime metric g_{ab} is very small, except near black holes.

We have argued that free particles (subject only to gravitational forces) move along geodesics. In the Newtonian limit of the geodesic equation we have shown how the Christoffel symbols $\Gamma^a{}_{bc}$ are associated with gravitational forces and, in turn, how the spacetime metric g_{ab} can be associated with the gravitational potential. However, we do not know yet what equation the metric g_{ab} has to satisfy. To motivate it, note that the gravitational potential in the Newtonian theory satisfies

$$\nabla^2 \phi = 4\pi G \rho, \tag{6.11}$$

where ρ is the mass density. The relativistic analogue of this equation should be tensorial and of second order in the derivatives of the metric. To take this analogy further, consider two neighbouring particles moving in a gravitational field with a potential ϕ with coordinates $x^i(t)$ and $x^i(t) + \xi^i(t)$ respectively, with $\xi^i(t)$ small and $i = 1, 2, 3$. The equations of motion are then given:

$$\ddot{x}^i = -\frac{\partial \phi(x)}{\partial x^i}$$

and

$$\ddot{x}^i + \ddot{\xi}^i = -\frac{\partial \phi(x)}{\partial x^i} - \xi^j \frac{\partial^2 \phi}{\partial x^i \partial x^j} + O(\xi^2).$$

Subtracting the two last equations:

$$\ddot{\xi}^i = -\xi^j \frac{\partial^2 \phi}{\partial x^i \partial x^j}.$$

This is the relative acceleration of two test particles separated by a 3-vector ξ^i – the second derivative of the potential gives the tidal forces. This is in analogy to the geodesic deviation equation:

$$\nabla_V \nabla_V \xi^a = R^a{}_{cdb} V^c V^d \xi^b,$$

provided that one identifies

$$-\xi^j \frac{\partial^2 \phi}{\partial x^i \partial x^j} \quad \text{and} \quad R^a{}_{cdb} V^c V^d \xi^b.$$

This identification would make clear the relation between gravity and geometry – note that the Riemann tensor involves second derivatives of the metric tensor.

6.2 Principles employed in General Relativity

The main idea underlying General Relativity is that matter –including energy– curves spacetime (assumed to be a smooth Lorentzian manifold). This in turn affects the motion of particles and light rays, postulated to move on timelike and null geodesics of the manifold, respectively. These ideas are understood in conjunction with the main principles of General Relativity, listed below.

(1) Equivalence Principle. In small enough regions of spacetime, the laws of physics reduce to those of Special Relativity; it is impossible to detect the existence of a gravitational field by means of *local* experiments.

(2) Principle of General Covariance. This states that laws of Nature should have the same mathematical form in any reference frame; hence, they should be tensorial.

(3) Principle of minimal gravitational coupling. This is used to derive the General Relativity analogues of Special Relativity results. According to this principle, one should change

$$\eta_{ab} \to g_{ab}, \quad \partial \to \nabla.$$

For example, in Special Relativity the equations for a perfect fluid are given by:

$$T^{ab} = (\rho + p) V^a V^b - p\, \eta^{ab},$$
$$\partial_a T^{ab} = 0.$$

In General Relativity these should be changed to:

$$T^{ab} = (\rho + p) V^a V^b - p\, g^{ab},$$
$$\nabla_a T^{ab} = 0.$$

(4) Correspondence principle. General relativity must agree with Special Relativity in absence of gravitation and with Newtonian gravitational theory in the case of weak gravitational fields and in the non-relativistic limit (slow speed).

6.2.1 The Einstein equations in vacuum

In vacuum, such as in the outside of a body in empty space, one has that the mass density ρ vanishes and the equation for the Newtonian potential becomes:

$$\nabla^2\phi = 0.$$

The Laplace equation involves an object with two indices, namely $\frac{\partial^2\phi}{\partial x^i \partial x^j}$. Therefore, one would guess that the gravitational field equations involve a symmetric geometric object with two indices, and hence the same number of components as the metric g_{ab}, arising from a contraction of the Riemann tensor (since the Riemann has two derivatives of the metric). The Ricci tensor is such a tensor and hence one would be tempted to guess that the gravitational field equations are

$$R_{ab} = 0. \tag{6.12}$$

These are indeed the correct equations for gravity in absence of matter fields and they are known as the *Einstein vacuum field equations*. The equations (6.12) form a set of ten nonlinear, second order partial differential equations for the components of the metric tensor g_{ab}. These are hard to solve, except simple settings with a high degree of symmetry.

Remark 1. One of the simplest solutions to the vacuum equations is the Minkowski metric. Expressing the metric g_{ab} locally as $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$, we see that all the Christoffel symbols vanish, from which $R_{ab} = 0$ is trivially satisfied.

Remark 2. The most general form of the vacuum equations which is tensorial and depends linearly on second derivatives of the metric is:

$$R_{ab} + \Lambda g_{ab} = 0,$$

where Λ is the so-called *Cosmological constant*. Outside Cosmology, Λ is usually taken to be zero.

6.2.2 The (full) Einstein Equations

Matter in relativity is described by a (0,2) tensor T_{ab} called the energy-momentum tensor. Therefore, in the presence of matter, one would be tempted to generalise (6.12) to

$$R_{ab} = \kappa\, T_{ab}$$

for some coupling constant κ. In fact, Einstein did suggest this equation. However, this equation is problematic for the following reason. The mass-energy is conserved and this is described by $\nabla^a T_{ab} = 0$, consistent with the minimal coupling principle that generalises of the equations motion in the Special Relativity case. However, in general $\nabla^a R_{ab} \neq 0$. Therefore, consistency with the conservation of mass-energy implies that we have to equate T_{ab} with a curvature tensor with vanishing divergence. There is only one (0,2) tensor, constructed from the Ricci tensor, which is automatically conserved: the Einstein tensor

$$G_{ab} = R_{ab} - \frac{1}{2} R\, g_{ab}$$

which always satisfies $\nabla^a G_{ab} = 0$. Therefore, one is led to propose

$$G_{ab} = \kappa\, T_{ab}.$$

as the field equation for the spacetime metric g_{ab} in the presence of matter-energy sources. Note, however, that since $\nabla^a g_{ab} = 0$, we could also have written

$$G_{ab} + \Lambda g_{ab} = \kappa T_{ab}. \tag{6.13}$$

These are the complete Einstein field equations for the metric g_{ab} of a spacetime.

Note that the Einstein equations are the simplest compatible with the Equivalence Principle, but they are not the only ones. In general, the Einstein field equations are extremely complicated set of non-linear partial differential equations. In some simple settings, analytic solutions may be found. These include:

(i) The vacuum spherically symmetric static case (the Schwarzschild spacetime).

(ii) The weak field case (gravitational waves).

(iii) The isotropic and homogeneous case (Cosmology).

We will study cases (i) and (ii) in the following sections.

6.2.3 Newtonian limit

To determine the value of the constant κ one needs to make contact with the Newtonian theory. In this subsection we are going to see how (6.13) (with $\Lambda = 0$) reproduces the Poisson equation for the gravitational potential in the Newtonian limit. From now on, we will set $\Lambda = 0$ unless otherwise stated.

Contracting both sides of (6.13) we find $R = -\kappa T$, which allows us to rewrite (6.13) as

$$R_{ab} = \kappa \left(T_{ab} - \tfrac{1}{2} T \, g_{ab} \right) \tag{6.14}$$

We want to show that this equation reduces to Newtonian gravity in the weak-field, time-independent and slowly moving limit. For simplicity, we consider dust as the source of energy-momentum, for which

$$T_{ab} = \rho \, U_a \, U_b \,,$$

where U^a is the dust four-velocity, and ρ is the energy density in the rest frame. The "dust" we are considering is a massive body, such as the Sun. Without loss of generality, we can work in the dust rest frame, in which

$$U^a = (U^0, 0, 0, 0) \,.$$

We can fix U^0 using the normalisation condition $g_{ab} U^a U^b = -1$. In the weak field limit, from (6.9) and (6.10) we can write

$$g_{00} = -1 + h_{00} \,, \quad g^{00} = -1 - h_{00} \,. \tag{6.15}$$

Then, to first order in h_{ab} we get

$$U^0 = 1 + \tfrac{1}{2} h_{00} \,.$$

In fact, we are already assuming that ρ is small. Therefore, the contributions from h_{00} to T_{ab} coming from the U_0 terms will be of higher order, and we can simply take $U^0 = 1$, and correspondingly $U_0 = -1$. Then,

$$T_{00} = \rho \,,$$

and all the other components of the stress-energy tensor T_{ab} vanish. In this limit, the rest energy $\rho = T_{00}$ will be much larger than the other terms in T_{ab}, so we can focus on the $a = b = 0$ component of (6.14). To the lowest non-trivial order, the trace of the energy momentum tensor is

$$T = g^{ab}T_{ab} = g^{00}T_{00} = -T_{00} = -\rho\,.$$

and hence, the 00-component of (6.14) becomes

$$R_{00} = \tfrac{1}{2}\,\kappa\,\rho\,. \tag{6.16}$$

Now we need to express the lhs of this equation in terms of the metric. To do so, we have to compute $R_{00} = R^a{}_{0a0} = R^i{}_{0i0}$. We have

$$R^i{}_{0j0} = \partial_j\Gamma^i{}_{00} - \partial_0\Gamma^i{}_{j0} + \Gamma^i{}_{ja}\Gamma^a{}_{00} - \Gamma^i{}_{0a}\Gamma^a{}_{j0}\,.$$

Note that the second term in this expression is a time derivative, which vanishes for static fields. The third and fourth terms are of the form $(\Gamma)^2$, and since the Christoffels Γ are of first order in the metric perturbation h_{ab}, these terms are of higher order and can be neglected. Therefore, to first order in h_{ab} we have $R^i{}_{0j0} = \partial_j\Gamma^i{}_{00}$. From this, we compute

$$\begin{aligned}
R_{00} &= R^i{}_{0i0} \\
&= \partial_i\left[\tfrac{1}{2}g^{ia}\left(\partial_0 g_{a0} + \partial_0 g_{0a} - \partial_a g_{00}\right)\right] \\
&= -\,\tfrac{1}{2}\,\delta^{ij}\partial_i\partial_j h_{00} \\
&= -\,\tfrac{1}{2}\nabla^2 h_{00}\,.
\end{aligned}$$

Then, equation (6.16) becomes

$$\nabla^2 h_{00} = -\kappa\,\rho\,. \tag{6.17}$$

From equation (6.9) we have $h_{00} = -2\,\phi$. Comparing with the Poisson equation for Newtonian gravity (6.11), we see that General Relativity does indeed reproduce Newtonian gravity if we set $\kappa = 8\pi\,G$, where G is Newton's gravitational constant.

Having fixed the normalisation correctly to reproduce the Newtonian limit we arrive at the final form of Einstein's equations for general relativity:

$$R_{ab} - \tfrac{1}{2}\,R\,g_{ab} = 8\pi\,G\,T_{ab}\,. \tag{6.18}$$

6.3 The Schwarzschild solution

In GR, the unique spherically symmetric vacuum solution of the Einstein equations is the Schwarzschild metric. It is second in importance only to Minkowski space and it corresponds to the static, spherically symmetric gravitational field in empty space surrounding a (spherically symmetric) source, such as a star. As we shall see later, it also represents a black hole.

The assumption of spherical symmetry and staticity severely constraints the form of the line element. Firstly, assuming that the spacetime is static means that there exists a timelike Killing vector field such that, far away from any sources, is of the form ∂_t, which is the canonical timelike Killing vector field in Minkowski space. Furthermore, in these coordinates the line element is invariant under a time inversion $t \to -t$. The assumption to preserve spherical symmetry implies that coordinates can be chosen such that the line element possesses an explicit round sphere, $d\Omega^2_{(2)} = d\theta^2 + \sin^2\theta\, d\phi^2$, where (θ, ϕ) are the standard angular coordinates on a unit 2-sphere. Therefore, with our symmetry assumptions, the most general line element that we can write down is of the following form:

$$\mathrm{d}s^2 = -e^{2A(r)}\mathrm{d}t^2 + e^{2B(r)}\mathrm{d}r^2 + r^2 e^{2C(r)}(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,. \tag{6.19}$$

We can use our freedom to choose the coordinates to simplify (6.19) further. Defining a new radial coordinate,

$$\bar{r} = r\, e^{C(r)} \quad \Rightarrow \quad d\bar{r} = \left(1 + r\,\tfrac{dC}{dr}\right) e^{C(r)}\, dr\,, \tag{6.20}$$

the metric (6.19) becomes

$$\mathrm{d}s^2 = -e^{2A(r)}\mathrm{d}t^2 + \left(1 + r\,\tfrac{dC}{dr}\right)^{-2} e^{2(B(r)-C(r))}\mathrm{d}\bar{r}^2 + \bar{r}^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,, \tag{6.21}$$

where r is now a function of $\bar{r}$ defined by (6.20). Making the following relabelings,

$$\bar{r} \to r\,, \quad \left(1 + r\,\tfrac{dC}{dr}\right)^{-2} e^{2(B(r)-C(r))} \to e^{2B(r)}\,,$$

the metric (6.21) becomes

$$\mathrm{d}s^2 = -e^{2A(r)}\mathrm{d}t^2 + e^{2B(r)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,. \tag{6.22}$$

This is the most general static and spherically symmetric spacetime. Note that in these coordinates, r has a physical meaning, namely is the areal radius of the 2-spheres.

Given the form of the metric (6.22), we are now ready to solve the Einstein vacuum equations,

$$R_{ab} = 0\,,$$

From (6.22), we find that the only non-vanishing components of the Ricci tensor are:

$$R_{tt} = e^{2(A-B)}\left(A'' + A'^2 - A'\,B' + \tfrac{2}{r}\,A'\right)\,, \tag{6.23}$$

$$R_{rr} = -A'' - A'^2 + A'\,B' + \tfrac{2}{r}\,B'\,, \tag{6.24}$$

$$R_{\theta\theta} = e^{-2B}\left[r(B' - A') - 1\right] + 1\,, \tag{6.25}$$

$$R_{\phi\phi} = \sin^2\theta\, R_{\theta\theta}\,, \tag{6.26}$$

where $'$ denotes $\frac{d}{dr}$. Having calculated the components of the Ricci tensor, we now have to equate them to zero. Since all components have to vanish independently, we can consider the combination

$$0 = e^{2(B-A)}R_{tt} + R_{rr} = \frac{2}{r}(A' + B')\,,$$

which implies $A(r) = -B(r) + c$, where c is a constant. We can set this constant to zero by rescaling the time coordinate by $t \to e^{-c}\,t$, after which we have

$$A(r) = -B(r)\,. \tag{6.27}$$

Considering $R_{\theta\theta} = 0$, using the previous result this equation now becomes

$$e^{2A}(2\,r\,A' + 1) = 1\,,$$

which is equivalent to

$$\partial_r\left(r\,e^{2A}\right) = 1\,.$$

This equation can be straightforwardly integrated to obtain

$$e^{2A(r)} = 1 - \frac{R_S}{r}\,, \tag{6.28}$$

where R_S is an undetermined constant. Using the results (6.27) and (6.28), we find that the spacetime metric that solves the Einstein vacuum equations is

$$\mathrm{d}s^2 = -\left(1 - \frac{R_S}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{R_S}{r}} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\,. \tag{6.29}$$

This metric depends on a single parameter, namely the constant R_S, which is called the Schwarzschild radius. To fix this constant in terms of a physical parameter, recall that in the weak field limit (i.e., far away from the source), the tt-component of the spacetime metric sourced by a mass M is given by

$$g_{tt} = -\left(1 - \frac{2GM}{r}\right)\,. \tag{6.30}$$

The metric (6.29) should reduce to the weak field case when $r \gg R_S$, but for the tt-component to agree with (6.30) we need to identify

$$R_S = 2GM\,. \tag{6.31}$$

We can now write down the final form of a static, spherically symmetric spacetime metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\,. \tag{6.32}$$

This line element is known as the *Schwarzschild metric*, and it depends on a single parameter, namely M. This parameter can be interpreted as the mass of the spacetime. Note that as $M \to 0$, we recover Minkowski space, as expected. Note also that as $r \to \infty$, the metric (6.32) becomes more like Minkowski space; this property is known as asymptotic flatness.

Remark 1. This solution demonstrates how the presence of mass curves flat spacetime.

Remark 2. The solution only applies to the exterior of a star, where there is vacuum. We will see shortly that, in the absence of matter, this solution describes a black hole.

Remark 3. *The Birkhoff Theorem:* a spherically symmetric solution in vacuum is necessarily static. That is, there is no time dependence is spherically symmetric solutions. Therefore, the assumption of staticity is not necessary.

Singularities

We see from (6.32) that the some of the metric coefficients become infinite or zero at $r = 0$ and $r = 2GM$, which suggests that something may be going wrong there. The metric coefficients are of course coordinate dependent; hence, it is entirely possible that the apparent problems at those values of the radial coordinate r are simply coordinate singularities that result in a breakdown of the coordinates rather than a problem with the spacetime manifold itself. For instance, this is precisely what happens at the origin of polar coordinates in flat space, where the metric $ds^2 = dr^2 + r^2 d\theta^2$ becomes degenerate and $g^{\theta\theta}$ blows up at $r = 0$. Of course, we know that there is nothing wrong with flat space at $r = 0$: this point is equivalent to any other point of the manifold, and by changing to Cartesian coordinates we see that both the metric $ds^2 = dx^2 + dy^2$ and its inverse are perfectly well-behaved at $x = y = 0$ ($r = 0$).

Therefore, in GR we need to assess singularities in a coordinate independent way. In general, this is difficult but for our present purposes we will identify singularities as places where the curvature of spacetime becomes infinite. The curvature is measured by the Riemann tensor, so to say that the curvature become infinite one cannot simply use the components of this tensor since they are coordinate-dependent. However, from the curvature one can construct scalars and, since the latter are coordinate independent, they provide a meaningful way to assess when the curvature becomes infinite. Scalars involving the Ricci scalar R or the Ricci tensor, e.g., $R_{ab}R^{ab}$, are not useful since they are fixed by the Einstein equations and, in the vacuum case, they trivially vanish. On the other hand, scalar quantities such as $R_{abcd}R^{abcd}$ or $R_{abcd}R^{cdef}R_{ef}{}^{ab}$ contain information about the curvature of the spacetime which is not determined by the Einstein equations and hence we can use them to detect physical singularities. If any of these scalars (but not necessarily all of them) blows up as we approach a certain point on the manifold, we regard that point as a singularity of the curvature. We should also check that this point is not infinitely far away in physical distance, that is, that it can be reached by observers or light travelling a finite distance along a curve.

Therefore, we have a sufficient condition for a point to be considered a singularity, but it is not a necessary condition. For the Schwarzschild metric (6.32), we find that

$$R_{abcd}R^{abcd} = \frac{48G^2M^2}{r^6}\,. \tag{6.33}$$

This scalar of the curvature, known as the Kretschmann scalar, blows up at $r = 0$, which is sufficient to convince us that $r = 0$ is a true singularity in the manifold. The other potentially troublesome point is $r = 2GM$, the Schwarzschild radius. We see that the Kretschmann scalar (6.33) (and in fact any other curvature scalar) is perfectly well-behaved there. This suggests that the singularity at $r = 2GM$ may just be a coordinate singularity and that the spacetime metric may be perfectly smooth there in more appropriate coordinates. We will see that this is indeed the case and that it is possible to find coordinates such that the Schwarzschild metric is smooth at $r = 2GM$; as we shall see, this surface corresponds to the event horizon of a black hole.

In the case of the Sun, it is a body that extends to a radius of $R_\odot = 10^6 GM_\odot$. Therefore, the surface $r = 2GM_\odot$ is far inside the Sun and hence the Schwarzschild metric does not apply there. On the other hand, there are compact objects for which the Schwarzschild metric is valid everywhere; as we will see, these objects are in fact black holes.

Remark 4. Uniqueness Theorem (Israel '67): The Schwarzschild metric (6.32) is the unique static, topologically spherical, asymptotically flat black hole solution of the Einstein vacuum equations.

6.4 Symmetries and Killing vectors

We think of a manifold M as possessing a symmetry if the geometry is invariant under a certain transformation that maps M onto itself; that is, if the metric is the same, in a suitable sense, from one point to another. Symmetries of the metric are called isometries. For example, consider four-dimensional Minkowski space

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2\,. \tag{6.34}$$

We already know of some isometries of this spacetime: translations $x^i \to x^i + a^i$, where a^i is a constant 4-vector, and Lorentz transformations $x^i \to \Lambda^i{}_j\, x^j$. In general, whenever a given metric g_{ab} is independent of a coordinate x^{c*}, i.e., $\partial_{c*} g_{ab} = 0$, there will be symmetry under translations along x^{c*}:

$$\partial_{c*} g_{ab} = 0\,, \quad \Rightarrow \quad x^{c*} \to x^{c*} + a^{c*} \text{ is a symmetry} \tag{6.35}$$

where a^{c*} is a constant.

Symmetries of the type (6.35) have a direct consequence for the motion of test particles as described by geodesics. Recall that the geodesic equation can be written in terms of the 4-momentum $p^a = m\, U^a$ (valid for timelike geodesics) as

$$p^a \nabla_a p^b = 0 \quad \Rightarrow \quad p^a \partial_a p_b - \Gamma^c{}_{ab}\, p^a\, p_c = 0\,. \tag{6.36}$$

The first term tells us how the momentum components change along the path

$$p^a \partial_a p_b = m\, \frac{dx^a}{d\tau}\, \partial_a p_b = m\, \frac{dp_b}{d\tau}\,, \tag{6.37}$$

while the second term gives

$$\begin{aligned} \Gamma^c{}_{ab}\, p^a\, p_c &= \tfrac{1}{2}\, g^{cd} (\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab})\, p^a\, p_c \\ &= \tfrac{1}{2} (\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab})\, p^a\, p^d \\ &= \tfrac{1}{2} (\partial_b g_{ad})\, p^a\, p^d\,. \end{aligned} \tag{6.38}$$

So, without making any symmetry assumptions, we see that the geodesic equation can be written as

$$m\, \frac{dp_b}{d\tau} = \tfrac{1}{2} (\partial_b g_{ad})\, p^a\, p^d\,. \tag{6.39}$$

Therefore, if all the metric components are independent of the coordinate x^{c*}, we find that this isometry implies that the component p_{c*} of the 4-momentum is conserved along the geodesic, i.e., it is a constant of motion:

$$\partial_{c*} g_{ab} = 0 \quad \Rightarrow \quad \frac{dp_{c*}}{d\tau} = 0 \tag{6.40}$$

This holds for any geodesics, including the null ones, even though we have derived it for the timelike ones only.

Notice that one could in principle transform a given metric into a complicated coordinate system in which the translational symmetries are not obvious anymore. The same is true for more complicated symmetries. Therefore, one needs a more systematic procedure to deal with symmetries. We can develop such a procedure by casting (6.40) in a manifestly covariant form. If x^{c*} is the coordinate which g_{ab} is independent of, consider the vector $\partial_{c*} \equiv \frac{\partial}{\partial x^{c*}}$, which we label as K:

$$K = \partial_{c*}\,, \tag{6.41}$$

which is equivalent in component notation to

$$K^a = (\partial_{c*})^a = \delta^a_{c*}\,. \tag{6.42}$$

We say that the vector K^a generates the isometry; this means that the transformation under which the geometry is invariant is expressed infinitesimally as a motion in the direction of K^a. In terms of this vector, the non-covariant looking quantity p_{c*} is simply

$$p_{c*} = K^a\, p_a = K_a\, p^a\,. \tag{6.43}$$

On the other hand, the constancy of this invariant quantity along the path is equivalent to the statement that its directional derivative along the geodesic vanishes:

$$\frac{dp_{c*}}{d\tau} = 0 \quad \Leftrightarrow \quad p^a \nabla_a (K_b\, p^b) = 0\,. \tag{6.44}$$

Expanding the expression on the right, we obtain

$$\begin{aligned} p^a \nabla_a (K_b\, p^b) &= p^a\, K_b\, \nabla_a p^b + p^a\, p^b\, \nabla_a K_b \\ &= p^a\, p^b\, \nabla_a K_b \\ &= p^a\, p^b\, \nabla_{(a} K_{b)}\,, \end{aligned} \tag{6.45}$$

where in the second line we have used the geodesic equation of motion $p^a \nabla_a p^b = 0$. Therefore, we conclude that any vector K_a that satisfies $\nabla_{(a} K_{b)} = 0$ implies that $K_b p^b$ is conserved along the geodesic:

$$\nabla_{(a} K_{b)} = 0 \quad \Rightarrow \quad p^a \nabla_a (K_b\, p^b) = 0\,. \tag{6.46}$$

The equation on the left is known as Killing's equation and vector fields that satisfy it are known as Killing vectors fields (or simply as Killing vectors). It can be easily shown that if the metric is independent of some coordinate x^{c*}, the vector ∂_{c*} will satisfy Killing's equation. In fact, if a vector K^a satisfies Killing's equation, it is always possible to find coordinates such that $K = \partial_{c*}$; however, in general one cannot find coordinates in which all Killing vectors are simultaneously of this form, nor is this form necessary for the vector to satisfy Killing's equation.

6.5 Geodesics of the Schwarzschild geometry

The classical experimental tests of General Relativity are based on the Schwarzschild solution. These are based on the comparison of the trajectories of freely falling particles and light rays in gravitational field of a central body with their counterparts in Newtonian theory. Therefore, we have to consider geodesics, both timelike and null, in the Schwarzschild geometry. This changed with the recent detection of gravitational waves by the LIGO/Virgo collaboration, but we will postpone the discussion of gravitational waves to the next chapter.

In order to derive the geodesics in Schwarzschild spacetime, it is best to use the Euler-Lagrange equations with

$$\mathcal{L} = -\left(1 - \frac{2GM}{r}\right)\dot{t}^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)\,, \tag{6.47}$$

where $\dot{}$ denotes differentiation with respect to an affine parameter λ. For timelike geodesics one has $\mathcal{L} = -1$, while for null geodesics $\mathcal{L} = 0$ (and for spacelike geodesics $\mathcal{L} = +1$).

The Euler-Lagrange equations for the geodesics are:

$$\ddot{t} + \frac{2GM}{r(r-2GM)}\,\dot{r}\,\dot{t} = 0\,, \tag{6.48a}$$

$$\ddot{r} + \frac{GM}{r^3}(r-2GM)\dot{t}^2 - \frac{GM}{r(r-2GM)}\,\dot{r}^2 - (r-2GM)\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) = 0\,, \tag{6.48b}$$

$$\ddot{\theta} + \frac{2}{r}\,\dot{\theta}\,\dot{r} - \sin\theta\,\cos\theta\,\dot{\phi}^2 = 0\,, \tag{6.48c}$$

$$\ddot{\phi} + \frac{2}{r}\,\dot{\phi}\,\dot{r} + 2\,\cot\theta\,\dot{\theta}\,\dot{\phi} = 0\,. \tag{6.48d}$$

Note that the Schwarzschild metric possesses 4 Killing vectors: three for the spherical symmetry, and one for time translations. Each of these Killing vectors leads to a constant of motion for the free particle (i.e., geodesics). Recall that if K^a is a Killing vector and $\dot{x}^a$ is the vector tangent to a geodesic, then

$$K_a\,\dot{x}^a = \text{constant} \tag{6.49}$$

along the geodesic. In addition, there is always another constant of motion for the geodesics: the Lagrangian itself. Indeed, the geodesic equation together with metric compatibility implies that the quantity

$$\mathcal{L} = g_{ab}\,\dot{x}^a\,\dot{x}^b$$

is constant along the geodesic.

Invariance under time translations leads to conservation of the energy, while invariance under spatial rotations leads to conservation of the three components of the angular momentum. The angular momentum can be thought of as a three vector with a magnitude (one component) and a direction (two components). Conservation of the direction of the angular momentum means that the particle will move in a plane. Because of the spherical symmetry of the Schwarzschild solution, without loss of generality we can choose this plane to be the equatorial plane in our coordinate system. Thus, the two Killing vectors that lead to conservation of the direction of the angular momentum imply that, for a single particle, we can choose

$$\theta = \frac{\pi}{2}\,.$$

The two remaining Killing vectors lead to the conservation of the energy and the magnitude of the angular momentum. The energy arises from the canonical timelike Killing vector

$$T^a = (\partial_t)^a \quad \Rightarrow \quad T_a = -\left(1 - \frac{2GM}{r}\right)(dt)_a\,.$$

Similarly, conservation of the angular momentum is associated to the canonical rotational Killing vector

$$R^a = (\partial_\phi)^a \quad \Rightarrow \quad R_a = r^2\,\sin^2\theta\,(d\phi)_a\,.$$

Since $\theta = \frac{\pi}{2}$ for the equatorial geodesics, we find that the two conserved quantities are

$$E = -T_a\,\dot{x}^a = \left(1 - \frac{2GM}{r}\right)\dot{t}\,, \tag{6.50}$$

$$L = R_a\,\dot{x}^a = r^2\,\dot{\phi}\,. \tag{6.51}$$

Indeed, since the Lagrangian (6.47) does not explicitly depend on either t or ϕ, the corresponding components of the Euler-Lagrange equations (6.48a) and (6.48d) in fact are

$$\frac{d}{d\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{t}}\right) = 0 \quad \Rightarrow \quad \frac{\partial\mathcal{L}}{\partial\dot{t}} = -2\left(1 - \frac{2GM}{r}\right)\dot{t} = \text{constant}\,, \tag{6.52}$$

$$\frac{d}{d\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) = 0 \quad \Rightarrow \quad \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = 2\,r^2\,\dot{\phi} = \text{constant}\,, \tag{6.53}$$

which results in the previous definitions of E and L.[2]

Having obtained the conserved quantities E and L, we can now get an understanding of the orbits of particles, both timelike and null, in the Schwarzschild spacetime. To do so, let us consider the Largrangian (6.47) that governs the geodesics specialised to trajectories on the $\theta = \frac{\pi}{2}$ plane:

$$-\left(1 - \frac{2GM}{r}\right)\dot{t}^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\dot{r}^2 + r^2\,\dot{\phi}^2 = -\epsilon\,, \tag{6.54}$$

where $\epsilon = 1, 0, -1$ for timelike, null and spacelike geodesics respectively. Multiplying by $\left(1 - \frac{2GM}{r}\right)$ and using the expressions for E and L, equations (6.50) and (6.51), we obtain,

$$-E^2 + \dot{r}^2 + \left(1 - \frac{2GM}{r}\right)\left(\frac{L^2}{r^2} + \epsilon\right) = 0 \tag{6.55}$$

Multiplying this equation by $\frac{1}{2}$, we arrive at the following expression

$$\tfrac{1}{2}\,\dot{r}^2 + V(r) = \mathcal{E}\,, \tag{6.56}$$

where

$$V(r) = \frac{1}{2}\,\epsilon - \epsilon\,\frac{GM}{r} + \frac{L^2}{2\,r^2} - \frac{GML^2}{r^3}\,, \tag{6.57}$$

and

$$\mathcal{E} = \tfrac{1}{2}\,E^2\,. \tag{6.58}$$

Equation (6.56) is precisely the equation of motion for a classical particle of unit mass and "energy" $\mathcal{E}$ moving in a potential $V(r)$. Note that while the conserved energy is E, the effective potential for the motion along the radial direction r is sensitive to $\mathcal{E} = \frac{E^2}{2}$. Our ultimate goal is to obtain the full trajectory of the particle, that is $r(\lambda)$ and $t(\lambda)$ and $\phi(\lambda)$. Nevertheless, as we shall see, understanding the radial motion provides very useful intuition about the actual orbits.

A similar analysis of the orbits of a particle of unit mass in Newtonian gravity would have led to a very similar result: we would have ended up with equation (6.56) but with a different effective potential; the effective potential for the radial motion in Newtonian gravity does not contain the last term in (6.57). In the potential (6.57), the first term is just a constant; the second term corresponds to the Newtonian gravitational potential, and the third term is the angular momentum contribution, which leads to a centrifugal repulsion. The last term is the GR contribution and it leads to differences in the motion compared to the Newtonian case, especially for small r.

[2] Note that E is not the energy of a particle with 4-momentum p^a as measured by an observer with 4-velocity U^a; the latter would be $-p_a U^a$. The reason why they do not agree is that U^a is normalised to $U_a U^a = -1$, while $T^a = (\partial_t)^a$ is not; only this form of T^a corresponds to a Killing vector.

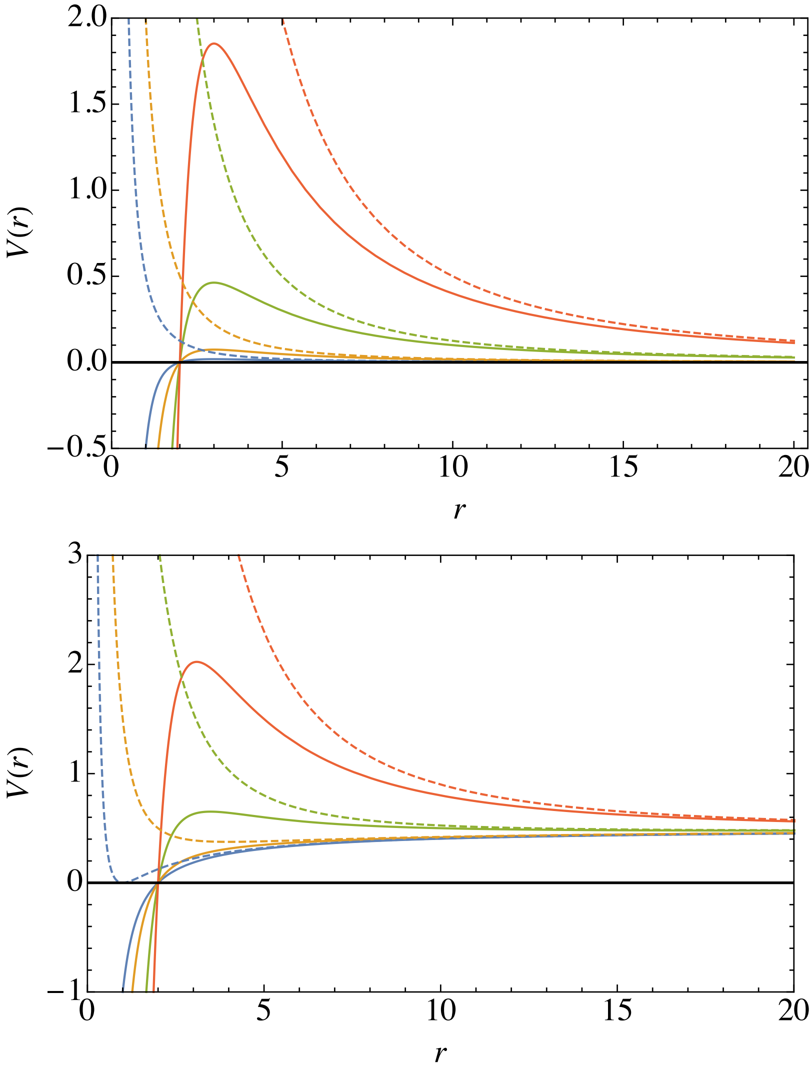


Figure 6.1: Effective potential for geodesics for different values of L. *Top*: null geodesics ($\epsilon = 0$). *Bottom*: timelike geodesics, i.e., trajectories of free massive particles ($\epsilon = 1$). The GR potential is corresponds to the solid curves while the Newtonian one corresponds to the dashes curves. The shown values are $L = 1$ (blue), 2 (orange), 5 (green), and 10 (red). We have chosen $GM = 1$.

The possible orbits can be determined by comparing $\mathcal{E}$ to $V(r)$ in Fig. 6.1 for the different values of L. This is so because $\mathcal{E}$ is conserved along the trajectory and equation (6.56) tells us that as the particle moves, both the kinetic energy (i.e., $\frac{1}{2}\dot{r}^2$) and the potential energy (i.e., $V(r)$) change in a way such that $\mathcal{E}$ remains constant. Therefore, starting from a given value of r such that $\dot{r} \neq 0$, the particle will move until it reaches a turning point where $V(r) = \mathcal{E}$, when it will start moving in the opposite direction. Depending on $\mathcal{E}$ and L, there may be no turning point, in which case the particle just keeps moving. For example, for a fixed L, this happens when $\mathcal{E}$ is larger than the maximum of $V(r)$. In other cases, the particle may describe a circular orbit at a constant $r = r_c$ (and hence $\dot{r} = 0$). This happens at an extremum of the potential, $\frac{dV}{dr} = 0$.

Differentiating (6.57), we find that circular orbits occur when

$$\epsilon GMr_c^2 - L^2 r_c + 3GML^2\,\gamma = 0\,, \tag{6.59}$$

where $\gamma = 0$ in Newtonian gravity and $\gamma = 1$ in GR. Circular orbits will be stable if they correspond to a minimum of the potential, and unstable if they correspond to a maximum. Bound orbits that are not circular will oscillate around the radius of a stable circular orbit and hence they will correspond to ellipses.

For Newtonian gravity, circular orbits occur at

$$r_c = \frac{L^2}{\epsilon GM}\,. \tag{6.60}$$

For massless particles, $\epsilon = 0$ and there are no circular orbits. This is consistent with the top plot in Fig. 6.1, which shows that the dashed curves have no extrema. Therefore, in Newtonian gravity, a photon with energy E shot from $r = \infty$ will move towards smaller values of r until it reaches a turning point and moves back to $r = \infty$. On the other hand, for massive there stable circular orbits at the radius (6.60) as well as bound orbits that oscillate around this radius. If the energy is greater than the asymptotic value $E = 1$, the orbits will be unbound and they will describe a particle that approaches a gravitating body but eventually escapes.

In GR the situation is different, especially at small values of r, where the term $-GML^2/r^3$ dominates $V(r)$. Indeed, for $r \to 0$ the potential goes to $-\infty$ in GR while it goes to $+\infty$ in Newtonian gravity. This is clearly seen in Fig. 6.1 comparing the solid curves (GR) with the dashed ones (Newton's gravity). In GR, $V(r) = 0$ at $r = 2GM$ for any value of L; inside this radius is the black hole, which we will discuss more thoroughly later. For massless particles, there is always a barrier (except for $L = 0$, for which the potential vanishes), but a sufficiently energetic photon (i.e., with $\mathcal{E}$ that is greater than the maximum of $V(r)$) will be able to overcome the barrier and inevitably fall to $r = 0$. At the top of the potential barrier there are unstable circular orbits. For $\epsilon = 0$ and $\gamma = 1$ equation (6.59) gives

$$r_c = 3GM\,. \tag{6.61}$$

A photon can orbit forever in a circle precisely at this radius, but any small perturbation will drive it to $r = 0$ or $r = \infty$. This radius is also known as the photon sphere, and whilst it is unstable for the Schwarzschild black holes, rotating black holes can have stable photon spheres, which play an important role in the recent images of supermassive black holes by the Event Horizon Telescope.[3]

For massive particles, equation (6.59) tells us that there are circular orbits at

$$r_c = \frac{L^2 \pm \sqrt{L^4 - 12G^2M^2L^2}}{2GM}\,. \tag{6.62}$$

For $L > \sqrt{12}GM$, there will be two circular orbits, one stable and one unstable. In the $L \to \infty$ limit, their radii are

$$r_c = \frac{L^2 \pm L^2(1 - 6G^2M^2/L^2)}{2GM} = \left(\frac{L^2}{GM}, 3GM\right)\,,$$

In this limit the stable circular orbit occurs far away, while the unstable one approaches $3GM$, similar to the massless case. As we decrease L, the two orbits come closer together and they coincide when the discriminant in (6.62) vanishes, which happens for

$$L = \sqrt{12}\,GM\,,$$

[3]See `https://eventhorizontelescope.org/`.

for which

$$r_c = 6GM\,, \tag{6.63}$$

and they disappear for smaller values of L. Therefore, $6GM$ is the smallest possible radius of a stable circular orbit for a massive particle in the Schwarzschild metric. There are also unbound orbits, which come from infinity and turn around, and bound but non-circular orbits, which oscillate around the stable circular orbits. These orbits, which are describe exact ellipses in Newtonian gravity, will no longer do so in GR. Finally, there orbits which come from $r = \infty$ and continue all the way to $r = 0$; this can happen if the energy is higher than the barrier, or for $L < \sqrt{12}GM$, when there is no barrier at all.

To summarise, we have found that for the Schwarzschild spacetime, there are stable circular orbits for $r > 6GM$ and unstable circular orbits for $3GM < r < 6GM$. It is important to remember that these orbits correspond to geodesics, i.e., free particles. There is nothing that prevents an accelerating particle from dipping below $3GM$ and escaping to infinity, as long as it stays above $r = 2GM$.

6.6 Experimental tests of General Relativity

Up until the detection of gravitational waves, most experimental tests of GR involve the motion of test particles in the solar system, and hence geodesics of the Schwarzschild metric.[4] Einstein suggested three tests: precession of perihelia, deflection of light and gravitational redshift.

6.6.1 Perihelion precession

The perihelion of an elliptical orbit is the point of closest approach to the centre of the ellipse; in our case, the centre is where the source of the gravitational field is, namely the Sun. The precession of perihelia reflects the fact that non-circular orbits in GR are not perfect ellipses; to a good approximation, they are ellipses that precesses. The idea is to obtain the radius of the orbit r as a function of the angular coordinate ϕ; for a perfect ellipse, $r(\phi)$ is periodic with period 2π, which reflects the fact that the perihelion occurs at the same position each orbit. In this subsection, we will see how GR introduces a small correction to $r(\phi)$ such that the orbit no longer closes, giving rise to a precession.

We start considering the radial equation of motion for a massive particle in the Schwarzschild geometry (6.56). To get an equation for $\frac{dr}{d\phi}$, we multiply it by

$$r^2\dot{\phi} = L \quad \Rightarrow \quad \frac{1}{\dot{\phi}^2} = \frac{r^4}{L^2}\,,$$

giving

$$\left(\frac{dr}{d\phi}\right)^2 + \frac{r^4}{L^2} - \frac{2GM}{L^2}\,r^3 + r^2 - 2GM\,r = \frac{2\mathcal{E}}{L^2}\,r^4\,. \tag{6.64}$$

To solve this equation, we define a new (dimensionless) variable

$$u = \frac{L^2}{GMr}\,. \tag{6.65}$$

[4] the fact that the Sun is rotating can be neglected because its rotational velocity is very small compared to the speed of light.

From (6.60) we see that a Newtonian circular orbit corresponds to $u = 1$. In terms of the new variable u, the equation of motion (6.64) becomes

$$\left(\frac{du}{d\phi}\right)^2 + \frac{L^2}{G^2M^2} - 2\,u + u^2 - \frac{2G^2M^2}{L^2}\,u^3 = \frac{2\mathcal{E}L^2}{G^2M^2}\,. \tag{6.66}$$

Differentiating again with respect to ϕ, we obtain a second order equation for $u(\phi)$:

$$\frac{d^2u}{d\phi^2} - 1 + u = \frac{3G^2M^2}{L^2}\,u^2\,. \tag{6.67}$$

In Newtonian gravity, the term on the right hand side would not be present, and the equation can be solved exactly for u. Here we will treat this term perturbatively. To do so, we expand the solution u as a Newtonian solution u_0 plus a small perturbation u_1,

$$u = u_0 + u_1\,, \quad \text{with} \quad u_1 \ll 1\,.$$

The zeroth-order part of (6.67) is then

$$\frac{d^2u_0}{d\phi^2} - 1 + u_0 = 0\,, \tag{6.68}$$

while the first-order part is

$$\frac{d^2u_1}{d\phi^2} + u_1 = \frac{3G^2M^2}{L^2}\,u_0^2\,. \tag{6.69}$$

The solution of the zeroth-order equation is given by

$$u_0 = 1 + e\,\cos\phi\,, \quad \Rightarrow \quad r = \frac{L^2}{GM(1 + e\,\cos\phi)}\,. \tag{6.70}$$

This is the well-known result for Newtonian gravity (first found by Kepler): it describes an ellipse with eccentricity e.[5]

Plugging the Newtonian solution to the first order equation (6.69) yields

$$\begin{aligned}\frac{d^2u_1}{d\phi^2} + u_1 &= \frac{3G^2M^2}{L^2}(1 + e\,\cos\phi)^2 \\ &= \frac{3G^2M^2}{L^2}\left[\left(1 + \tfrac{1}{2}e^2\right) + 2\,e\,\cos\phi + \tfrac{1}{2}\,e^2\,\cos 2\phi\right]\,.\end{aligned} \tag{6.71}$$

To solve this equation, note

$$\begin{aligned}&\frac{d^2}{d\phi^2}(\phi\,\sin\phi) + \phi\,\sin\phi = 2\cos\phi\,, \\ &\frac{d^2}{d\phi^2}(\cos 2\phi) + \cos 2\phi = -3\cos 2\phi\,.\end{aligned} \tag{6.72}$$

Comparing with (6.71), we see that a solution to this equation is

$$u_1 = \frac{3G^2M^2}{L^2}\left[\left(1 + \tfrac{1}{2}\,e^2\right) + e\,\phi\,\sin\phi - \tfrac{1}{6}\,e^2\,\cos 2\phi\right]\,. \tag{6.73}$$

[5] An ellipse can be specified by the semi-major axis a, namely the distance from the centre to the furthest point on the ellipse, and the semi-minor axis b, which is the distance from the centre to the closes point. The eccentricity is then defined as $e^2 = 1 - \frac{b^2}{a^2}$.

The first term corresponds to a shift, whilst the third term oscillates around zero. Therefore, none of these two terms significantly alters the Newtonian solution, at least qualitatively. On the other hand, the second can have an important effect because it accumulates over successive orbits. This type of term is often known as a "secular" term. Combining this term with the zeroth order solution we find

$$u = 1 + e\,\cos\phi + \frac{3G^2M^2e}{L^2}\,\phi\,\sin\phi\,. \tag{6.74}$$

This is not the full solution, even to the perturbed equation, but it captures the key aspect of the modifications introduced by GR. In particular, note that this equation can be written as the equation for an ellipse but with an angular period which is *not* 2π:

$$u = 1 + e\,\cos[(1-\alpha)\phi]\,, \tag{6.75}$$

where we have defined

$$\alpha = \frac{3G^2M^2}{L^2}\,. \tag{6.76}$$

The equivalence of (6.74) and (6.75) can be checked by expanding $\cos[(1-\alpha)\phi]$ in a Taylor series for small α:

$$\begin{aligned}\cos[(1-\alpha)\phi] &= \cos\phi + \alpha\,\frac{d}{d\alpha}\cos[(1-\alpha)\phi]\bigg|_{\alpha=0}\,,\\ &= \cos\phi + \alpha\,\phi\,\sin\phi\,.\end{aligned}$$

Equation (6.75) implies that, during each orbit, the perihelion changes by some amount. Indeed, from (6.75) we see that the periodicity of the solution is

$$\Delta\phi = \frac{2\pi}{1-\alpha} \approx 2\pi(1+\alpha) \tag{6.77}$$

Therefore, during each orbit the perihelion advances by an angle

$$\delta\phi = 2\pi\alpha = \frac{6\pi G^2M^2}{L^2}\,. \tag{6.78}$$

We can convert L to more familiar quantities using the expressions valid for Newtonian orbits since the quantity we are considering is a small perturbation. The equation for an ellipse can be written as

$$r = \frac{(1-e^2)a}{1+e\,\cos\phi}\,, \tag{6.79}$$

where a is the semi-major axis. Comparing to the zeroth-order solution (6.70) we see that

$$L^2 \approx GM(1-e^2)a\,.$$

Plugging this into (6.78) and restoring explicit factors of the speed of light, we get

$$\delta\phi = \frac{6\pi GM}{c^2(1-e^2)a}\,. \tag{6.80}$$

Historically, the precession of Mercury was the first test of GR. In fact, Mercury's precession had been known since the mid XIX$^{\text{th}}$ century and it was in contradiction with the predictions of Newtonian gravity. Einstein knew of this discrepancy, and one of the first things that he did after formulating his theory of gravity was to calculate the modifications to Mercury's orbit introduced by GR. To his delight, he found out that Mercury's precession was accounted for in GR.[6] In the case of Mercury, the shift is $\delta\phi = 43.0$"/century where " stands for arcseconds.

Remark 1. The effect is largest in Mercury because of its high eccentricity and small period which results in a large shift.

Remark 2. For Venus one has a predicted shift of $8.6''$ and an observed of $8.4'' \pm 4.8''$. For the Earth one has $3.8''$ and $5.0'' \pm 1.2''$. For the asteroid Icarus $10.3''$ and $9.8'' \pm 0.8''$.

6.6.2 Bending of light

In this subsection we are going to consider the deflection of light rays that travel near a massive body, e.g., the Sun. Therefore, we are going to consider null geodesics. We proceed as in the previous subsection by re-writing the equation of motion (6.56) (with $\epsilon = 0$ in $V(r)$) as an equation for $\frac{dr}{d\phi}$. Defining a new variable $u = 1/r$, we get

$$\frac{d^2u}{d\phi^2} + u = \frac{3GMu^2}{c^2}\,. \tag{6.81}$$

As before, the term on the right hand side is the small correction introduced by GR and therefore, $\delta \equiv GM/c^2$ is small relative to u. We expand u into a Newtonian solution plus a small correction

$$u = u_0 + u_1$$

so that the zeroth-order part of (6.81) is

$$\frac{d^2u_0}{d\phi^2} + u_0 = 0\,, \tag{6.82}$$

and the first-order part is

$$\frac{d^2u_1}{d\phi^2} + u_1 = 3\,\delta\,u_0^2\,. \tag{6.83}$$

The solution to the zeroth-order equation (6.82) is

$$u_0 = \tfrac{1}{b}\sin\phi \quad \Rightarrow \quad r\sin\phi = b\,, \tag{6.84}$$

where b is a constant known as the impact parameter. This solution represents a straight line corresponding to a photon sent from $r = \infty$ (for $\phi = 0$) and that returns to $r = \infty$ ($\phi = \pi$) and whose distance of closest approach to the source is given by b. Therefore, the change in the angle ϕ along the trajectory is $\Delta\phi = \pi$.

Plugging the zeroth order solution (6.84) into the first order equation (6.83), we get

$$\frac{d^2u_1}{d\phi^2} + u_1 = \frac{3\delta}{b^2}\sin^2\phi\,. \tag{6.85}$$

This equation has the particular solution

$$u_1 = \frac{\delta}{b^2}(1 + C\,\cos\phi + \cos^2\phi)\,, \tag{6.86}$$

where C is an arbitrary integration constant. Therefore, the full solution up to first order corrections is

$$u = \frac{1}{b}\sin\phi + \frac{\delta}{b^2}(1 + C\,\cos\phi + \cos^2\phi)\,. \tag{6.87}$$

[6] Note that Mercury, being the closest planet to the Sun, is there the GR effects are larger.

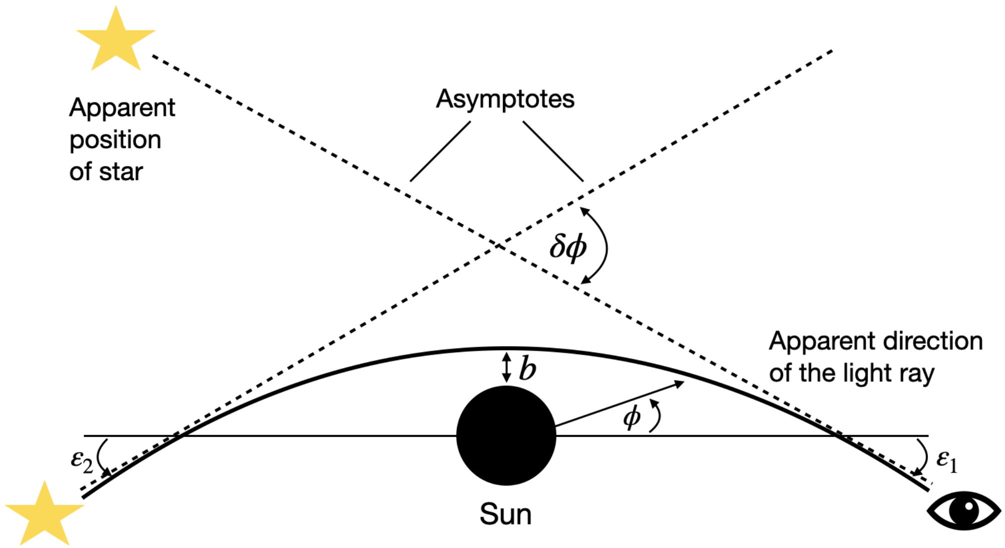


Figure 6.2: Deflection of light in a gravitational field.

We see from this solution that the effects of the GR corrections (the last two terms in this expression) is to make light deflect from a straight line. We are interested in determining the deflection angle $\delta\phi$ for a light ray in the presence of a spherically symmetric source, e.g., the Sun. Far away from the source, $r \to \infty$ and hence $u \to 0$, which requires the right hand side of (6.87) to vanish. Without loss of generality, lets take values of ϕ such that for $r \to \infty$ this angle asymptotes to $-\varepsilon_1$ and $\pi + \varepsilon_2$ respectively, see Fig. 6.2. Expanding (6.87) for small ε_1 and ε_2, we find

$$-\frac{\varepsilon_1}{b} + \frac{\delta}{b^2}(2 + C) = 0\,, \qquad -\frac{\varepsilon_2}{b} + \frac{\delta}{b^2}(2 - C) = 0\,. \tag{6.88}$$

Adding these two equations we find the total deflection angle:

$$\begin{aligned} \delta\phi = \varepsilon_1 + \varepsilon_2 &= \frac{4\delta}{b} \\ &= \frac{4GM}{c^2\, b}\,. \end{aligned} \tag{6.89}$$

For a light ray just grazing the Sun this predicts a deflection of $1.75''$ which compares well with some recent radio observations yielding $\Delta = 1.73'' \pm 0.05''$.

Remark. This is the second famous test of General Relativity —more generally referred to as *bending of light*. A first measurement was carried out by Eddington and collaborators in 1919.

6.6.3 Gravitational redshift

Consider an observer with four-velocity U^a who is stationary in the Schwarzschild coordinates, $U^i = 0$. We could allow the observer to be moving, but this would merely superimpose a conventional Doppler shift to the gravitational redshift. Since the four-velocity of any observer satisfies $U_a U^a = -1$, for a stationary observer in the Schwarzschild geometry this implies

$$U^0 = \left(1 - \frac{2GM}{r}\right)^{-\frac{1}{2}}\,. \tag{6.90}$$

Such an observer measures the frequency of a photon following a null geodesic $x^a(\lambda)$ to be

$$\omega = -g_{ab}\,U^a\,\dot{x}^b\,,$$

since $\dot{x}^a$ is the null vector tangent to the geodesic. This relation defines the normalisation of the affine parameter λ. Therefore, we have

$$\begin{aligned}\omega &= \left(1-\frac{2GM}{r}\right)^{\frac{1}{2}}\,\dot{t}\\ &= \left(1-\frac{2GM}{r}\right)^{-\frac{1}{2}}\,E\,.\end{aligned}\tag{6.91}$$

where E has been defined in (6.50). Since E is conserved along the geodesic, then ω will take different values when measured at different values of the radial coordinate r. For a photon emitted at r_1 and observed at r_2, the measured frequencies are related by

$$\frac{\omega_2}{\omega_1} = \left(\frac{1-2GM/r_1}{1-2GM/r_2}\right)^{\frac{1}{2}}\,.\tag{6.92}$$

This is an exact result for the frequency shift; in the limit $r \gg 2GM$ we have

$$\begin{aligned}\frac{\omega_2}{\omega_1} &= 1-\frac{GM}{r_1}+\frac{GM}{r_2}\\ &= 1+\Phi_1-\Phi_2\,,\end{aligned}\tag{6.93}$$

where $\Phi = -GM/r$ is the Newtonian potential. This result shows that the frequency decreases as Φ increases, which happens as we climb out of a gravitational field and hence a redshift.

Remark. It used to be thought that gravitational redshift also constituted a test of General Relativity, but it turns out that any other theory compatible with the Equivalence Principle will predict a redshift.

The gravitational redshift was first detected in 1960 by Pound and Rebka, using gamma rays travelling upward a distance of 22 meters (the height of the physics building at Harvard). Subsequent tests have become increasingly more precise, often using aircraft and atomic clocks. In all cases, agreement with Einstein's theory has been found.

These are the three classic tests of GR proposed by Einstein. Since then, other tests of GR have been proposed, including the binary pulsar (to be discussed in the context of gravitational waves) and the gravitational time delay discovered and observed by Shapiro.

6.7 Black holes

In this section we will study objects that are described by the Schwarzschild metric for all radii, even $r < 2GM$. As we will see, this spacetime describes a black hole.

One way to understand the geometry of a spacetime is to determine its causal structure and for that we consider the light cones. It is sufficient to consider radial null geodesics, i.e., light rays moving in the radial direction. For such null geodesics, θ and ϕ are constant and we are left with

$$0 = ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2+\left(1-\frac{2GM}{r}\right)^{-1}dr^2\,,\tag{6.94}$$

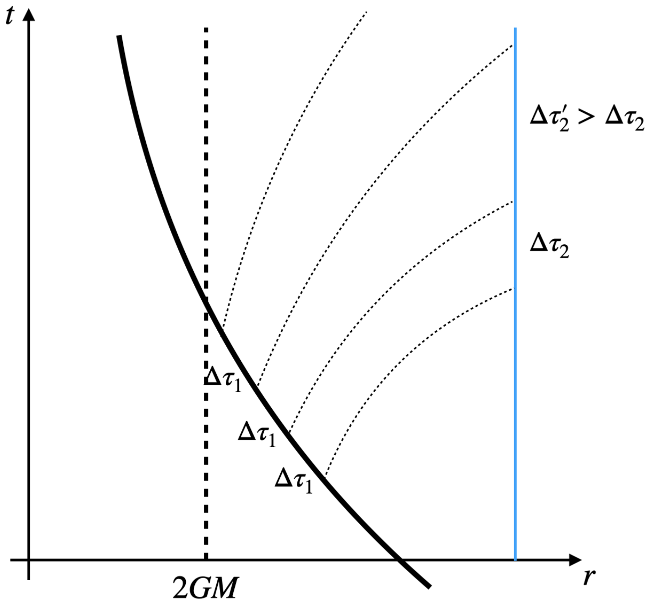


Figure 6.3: An observer falling into a black hole sends signals at intervals of proper time $\Delta\tau_1$. Another observer at a radius $r > 2GM$ receives the signals at successively longer intervals $\Delta\tau_2' > \Delta\tau_2$.

from which it follows that

$$\frac{dt}{dr} = \pm\left(1 - \frac{2GM}{r}\right)^{-1} . \tag{6.95}$$

This expression corresponds to the slopes of the light cones in the (t, r) plane. For large r, the slopes are ± 1, just as in flat space. However, as we approach $r = 2GM$, we have $\frac{dt}{dr} \to \pm\infty$, which implies that, in these coordinates, the light cones close up. Therefore, a light ray that approaches $r = 2GM$ never seems to get there. As we will see, this apparent inability to get to $r = 2GM$ is just an illusion caused by the fact that there is a coordinate singularity at $r = 2GM$; a light ray (or a massive particle) has no trouble reaching this radius and continuing to smaller radii. However, a far away observer would not be able to tell. In other words, if an observer falls towards smaller radii and sends signals as she progresses, external observers far away would see the signals more and more slowly, see Fig. 6.3.

The fact that an observer at $r > 2GM$ never sees the infalling observer reach $r = 2GM$ is a physical statement. However, to determine whether the infalling observer can reach $r = 2GM$ (and beyond) in finite proper time, we need to change to coordinates that are well-behaved at $r = 2GM$. The problem with the current coordinates is that $\frac{dt}{dr} \to \infty$ along radial null geodesics that approach $r = 2GM$, so progress in the r direction becomes slower and slower compared to the time coordinate. We can fix this problem as follows. We can integrate the equation (6.95) characterising radial null geodesics to find

$$t = \pm r_* + \text{constant} \tag{6.96}$$

where r_* is the so called tortoise coordinate and it is given by[7]

$$dr_* = \frac{dr}{1 - \frac{2GM}{r}} \quad \Rightarrow \quad r_* = r + 2GM \ln\left(\frac{r}{2GM} - 1\right) . \tag{6.97}$$

[7]The tortoise coordinate r_* is only well-defined for $r \geq 2GM$

In terms of the tortoise coordinate the Schwarzschild metric becomes

$$ds^2 = \left(1 - \frac{2GM}{r}\right)(-dt^2 + dr_*^2) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,. \tag{6.98}$$

where r should be regarded as a function of r_*. Notice that in these coordinates, the light cones no longer close up but the metric is still singular at $r = 2GM$. In fact, in these coordinates, the surface $r = 2GM$ has been pushed to $r_* = -\infty$.

To proceed, we define coordinates that are adapted to (radial) null geodesics. Defining,

$$v = t + r_*\,, \tag{6.99}$$

$$u = t - r_*\,, \tag{6.100}$$

then infalling radial null geodesics are characterised by $v =$ constant, while outgoing null geodesics are given by $u =$ constant. Considering the Schwarzschild metric in the original coordinates (6.32) and changing coordinates as

$$t = v - r_* \quad \Rightarrow \quad dt = dv - \frac{dr}{1 - \frac{2GM}{r}}\,,$$

we get,

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dv^2 + 2\,dv\,dr + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,. \tag{6.101}$$

These coordinates are known as ingoing Eddington-Finkelstein coordinates and, in these coordinates, the metric (6.101) is manifestly regular (and invertible) at $r = 2GM$. Therefore we conclude that the apparent singularity in (6.32) at $r = 2GM$ is a mere coordinate singularity. In the ingoing Eddington-Finkelstein coordinates (6.101), radial null geodesics are given by

$$\text{ingoing}: \ \frac{dv}{dr} = 0\,, \qquad \text{outgoing}: \ \frac{dv}{dr} = \frac{2}{1 - \frac{2GM}{r}}\,. \tag{6.102}$$

In these coordinates, light cones are well-behaved at $r = 2GM$, and this surface is at a finite coordinate value. In fact, both ingoing null and timelike geodesics can go past this surface. Note that for null geodesics this is straightforward to see: along an infalling null geodesic $v =$ constant, the radial coordinate varies from say $r = \infty$ to $r = 0$. However, light cones tilt over at $r = 2GM$ since $\frac{dv}{dr}$ for the outgoing null geodesics changes sign there; inside this surface, all future directed paths go towards smaller values of r, see Fig. 6.4. Some null cones and radial null geodesics are indicated in Fig. 6.5. Surfaces of $t =$ const. are also shown; one sees that t becomes infinite on the surface $r = 2M$.

The surface $r = 2GM$ is a point of no return: once a test particle dips below it, it can never escape. A surface past which particles can never escape to infinity is defined as the event horizon of the black hole. In the Schwarzschild space time, the event horizon is located at $r = 2GM$. The event horizon is a null surface so it is really the causal structure of the spacetime that makes it impossible to cross the horizon in an outward going direction. Since nothing can escape the event horizon, thus the name black hole. A black hole is simply a region of spacetime separated from infinity by an event horizon. The notion of an event horizon is a global one; the location of the horizon is a statement about the spacetime as a whole and cannot be determined by the local geometry.

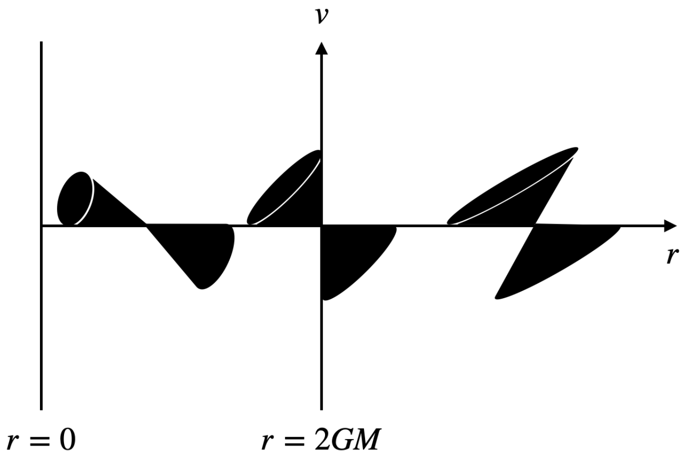


Figure 6.4: Light cones in the Schwarzschild geometry in ingoing Eddington-Finkelstein coordinates (v, r). For $r > 2GM$, outgoing null rays point towards larger values of r. On the other hand, for $r < 2GM$, both ingoing and outgoing light rays point towards smaller values of r.

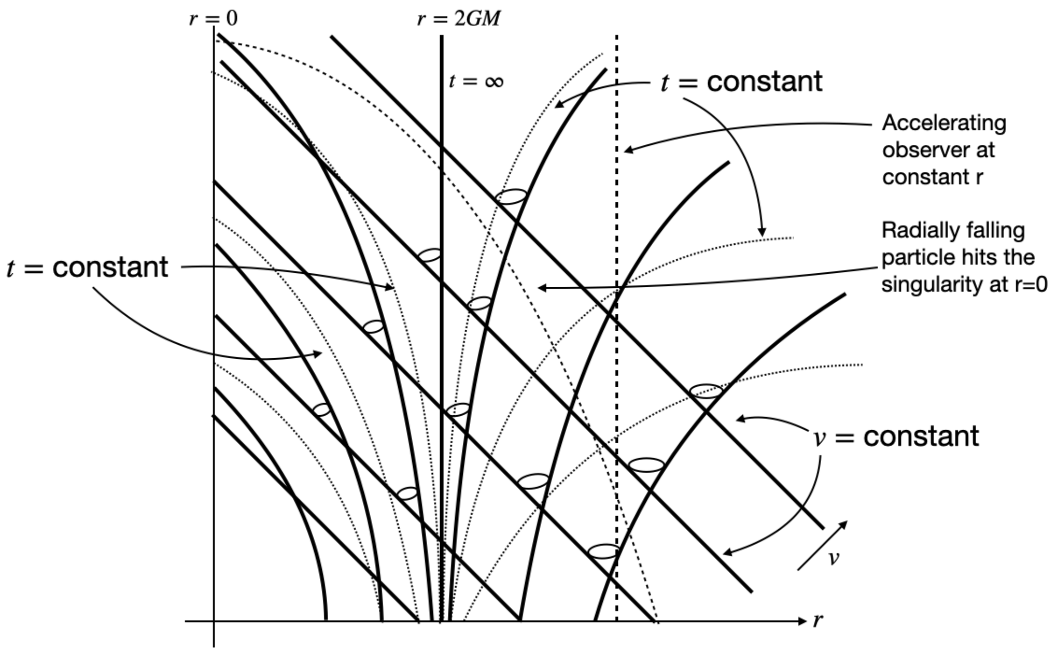


Figure 6.5: Finkelstein diagram in the (v, r) coordinates. Lines at 45° are lines of constant v. The surface $r = 2GM$ is a null surface on which $t = \infty$.

Consider a general static (and spherically symmetric) spacetime of the form,

$$ds^2 = -f(r)\, dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,, \tag{6.103}$$

where $f(r)$ vanishes at some $r = r_+$ with $r_+ > 0$, so $f(r_+) = 0$.[8] In this case one can show that $r = r_+$ is just a coordinate singularity of (6.103) corresponding to the location

[8]For simplicity we further assume that $f'(r_+) \neq 0$.

of a horizon. We can find regular coordinates there by considering the transformation to general ingoing Eddington-Finkelstein coordinates (v,r) as

$$dt = dv - \frac{dr}{f(r)}\,. \tag{6.104}$$

Then (6.103) becomes

$$ds^2 = -f(r)\,dv^2 + 2\,dv\,dr + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\,. \tag{6.105}$$

This line element is regular and invertible at $r = r_+$.

Maximally extended Schwarzschild spacetime

In the previous section we have shown that there exist coordinates (v,r) adapted to the infalling null geodesics such that we can go through the event horizon without encountering any problems. In fact, a local observer crossing the horizon may not even notice it since the local geometry at the horizon is no different from anywhere else. Therefore, $r = 2GM$ is a coordinate singularity in the original metric (6.32), and the region $r \leq 2GM$ should be included in our spacetime since physical particles can reach this region.

We will see now that we extend the spacetime in other directions. In the (v,r) coordinates, we can cross the horizon along future-directed curves, but not on past-directed ones. This looks suspicious since we started off with a time-symmetric spacetime $t \leftrightarrow -t$, and so reversing time should be a symmetry. We can see that this is indeed the case if we choose the coordinate u adapted to the outgoing null geodesics , (6.100), to write down the metric:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) du^2 - 2\,du\,dr + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\,. \tag{6.106}$$

In these coordinates, we can pass through the event horizon, but this time only along past directed null curves.

This may seem confusing: we can follow either future-directed or past-directed curves through the event horizon at $r = 2GM$, but we arrive at different places. To see this, note that from (6.99)–(6.100), if we keep $v =$ constant and decrease r we must have $t \to +\infty$, while if we keep u constant and decrease r we must have $t \to -\infty$ (since $r_* \to -\infty$ as $r \to 2GM$). Therefore, we have extended the original Schwarzschild spacetime (6.32) in two different directions: one to the future and one to the past.

At this stage, one may suspect that the spacetime can be further extended. Indeed, we can use (u,v) as our coordinates which leads to

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) du\,dv + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\,, \tag{6.107}$$

where r is defined implicitly in terms of u and v as

$$\tfrac{1}{2}(v-u) = r + 2GM\,\ln\left(\frac{r}{2GM} - 1\right)\,. \tag{6.108}$$

In these coordinates however, $r = 2GM$ is "infinitely far away" at either $v \to -\infty$ or $u \to +\infty$. To bring this surface at a finite coordinate distance, we can define new coordinates (U,V) as

$$U = -e^{-u/(4GM)}\,, \quad V = e^{v/(4GM)}\,, \tag{6.109}$$

which in terms of the original (t, r) coordinates are given by

$$U = -\left(\frac{r}{2GM} - 1\right)^{\frac{1}{2}} e^{-(t-r)/(4GM)}\,, \quad V = \left(\frac{r}{2GM} - 1\right)^{\frac{1}{2}} e^{(t+r)/(4GM)}\,. \tag{6.110}$$

In these coordinates, the Schwarzschild metric becomes

$$ds^2 = -\frac{32\, G^3 M^3}{r}\, e^{-r/(2GM)}\, dU\, dV + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,, \tag{6.111}$$

where r is defined implicitly by

$$U\, V = -\left(\frac{r}{2GM} - 1\right) e^{r/(2GM)}\,. \tag{6.112}$$

In the form of the metric (6.111), it is clear that nothing special happens at $r = 2GM$, and the metric is manifestly regular there.

The coordinates (U, V) are null coordinates in the sense that $\frac{\partial}{\partial U}$ and $\frac{\partial}{\partial V}$ are null vectors. We may get further intuition about the spacetime defining new coordinates such that one is timelike and the other one is spacelike. Therefore, we define

$$T = \frac{1}{2}(U + V) = \left(\frac{r}{2GM} - 1\right)^{\frac{1}{2}} e^{r/(4GM)} \sinh\left(\tfrac{t}{4GM}\right)\,, \tag{6.113}$$

$$R = \frac{1}{2}(V - U) = \left(\frac{r}{2GM} - 1\right)^{\frac{1}{2}} e^{r/(4GM)} \cosh\left(\tfrac{t}{4GM}\right)\,, \tag{6.114}$$

in terms of which the metric becomes

$$ds^2 = \frac{32\, G^3 M^3}{r}\, e^{-r/(2GM)}(-dT^2 + dR^2) + (d\theta^2 + \sin^2\theta\, d\phi^2)\,, \tag{6.115}$$

where r is implicitly defined by

$$T^2 - R^2 = \left(1 - \frac{r}{2GM}\right) e^{r/(2GM)}\,. \tag{6.116}$$

The coordinates (T, R, θ, ϕ) are known as Kruskal-Szekeres coordinates or Kruskal coordinates for short.

From (6.115), it is obvious that in Kruskal coordinates, radial null geodesics look like they do in flat space:

$$T = \pm R + \text{constant} \tag{6.117}$$

Furthermore, from (6.116) one can see that the event horizon $r = 2GM$ corresponds to

$$T = \pm R\,, \tag{6.118}$$

consistent with it being a null surface. More generally, from (6.116) we see that surfaces of constant r appear as hyperbolae in the $T - R$ plane,

$$T^2 - R^2 = \text{constant}\,. \tag{6.119}$$

On the other hand, from (6.113)–(6.114) we see that surfaces of constant t are given by

$$\frac{T}{R} = \tanh\left(\frac{t}{4GM}\right)\,, \tag{6.120}$$

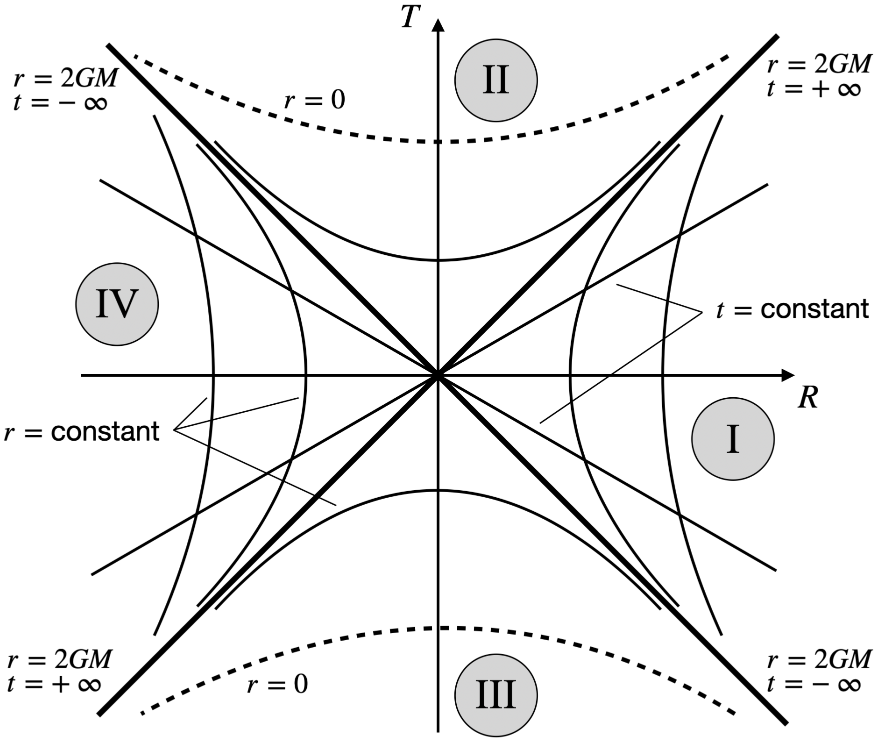


Figure 6.6: Kruskal diagram of the Schwarzschild spacetime. Null geodesics are straight lines at $\pm 45°$. Each point on this diagram is a round sphere of radius r. Regions I and IV are asymptotically flat regions, while region II is a black hole and region III is a white hole. $r = 0$ is a physical singularity and the manifold ends there.

which defines straight lines through the origin with slope $\tanh\left(\frac{t}{4GM}\right)$. Note that for $t \to \pm\infty$ (6.120) reduces to (6.118); therefore, $t = \pm\infty$ represents the same surface as $r = 2GM$.

The coordinates (T, R) should be allowed to range over every value they can take without hitting the real singularity at $r = 0$. Therefore, the allowed range of these coordinates is

$$-\infty \leq R \leq \infty\,, \qquad T^2 < R^2 + 1\,. \tag{6.121}$$

From (6.113) and (6.114) it may seem that T and R become imaginary for $r < 2GM$, but this is just an illusion caused by the fact that the (t, r) coordinates are not valid in this region. We can now draw spacetime diagram in the T-R plane (suppressing the angles on the two-sphere θ and ϕ), known as a Kruskal diagram, see Fig. 6.6. Each point on this diagram is a two sphere of radius r. This diagram represents the maximal analytic extension of the Schwarzschild spacetime; the coordinates cover the entire manifold described by this solution.

The original (t, r) coordinates were only valid for $r > 2GM$, which is only a portion of the Kruskal diagram. It is convenient to divide the diagram into four regions, as shown in Fig. 6.6. Region I corresponds to $r > 2GM$ and it is the region covered by the original (t, r) coordinates. By following future-directed null rays we reach region II, and by following past-directed null rays we reach region III. If we had studied spacelike geodesics we would have discovered region IV. The definitions (6.113) and (6.114) that relate the (T, R) coordinates to the original (t, r) ones are only valid in region I; in the other regions, one has to introduce the appropriate signs so that the coordinates remain real.

Now we describe the physical significance of the various regions in the Kruskal diagram. Region II is the black hole. Any causal curve (i.e., timelike or null) that travels from region I into II cannot go back. In fact, every future directed causal curve that enters region II will reach the singularity $r = 0$ in finite proper time. Therefore, any observer that falls into the black hole is doomed. That is, not only the observer that falls into the black hole cannot escape but also he/she inevitably has to move towards smaller r since this is a timelike direction. Indeed, in the original (t, r) coordinates, we see that for $r < 2GM$, t becomes spacelike and r becomes timelike. Thus, one cannot stop moving towards the singularity because we cannot stop the flow of time. Since proper time maximises the length of the geodesics, the observers that do not struggle against the inevitable fate (i.e., free falling observers) will live the longest, but they too will inevitably hit the singularity. The approach to the singularity is not particularly pleasant since the tidal forces become infinite. As the observer falls to the singularity the feet and the head will be pulled apart from each other, and the torso will be squeezed to infinitesimal distance. In fact, all the atoms in the body of the unfortunate observer will be torn apart by the infinite tidal forces. To summarise, ultimate death and destruction is what awaits the observer that falls into a black hole.

Region III is simply the time-reverse of region II, a part of the spacetime from which things can escape but they can never go back there. This region is known as the white hole. There is a singularity in the past at $r = 0$, out of which the universe springs. The boundary of region III is the past event horizon, while the boundary of region II is the future event horizon. Region IV cannot be reached from region I by any causal curve; likewise, no causal curve from region IV can reach region I. Region IV is another asymptotically flat region, which is a mirror image of region I. It can be thought of as being connected to region I by a wormhole (also known as an Einstein-Rosen bridge), a neck-like configuration joining to asymptotically flat regions.

In order to better understand the causal structure of the spacetime, it is convenient to collapse the whole Kruskal diagram into a finite region by constructing the conformal diagram, also known as Penrose diagram. Starting with the null version of the Kruskal coordinates (U, V) defined in (6.110), one can bring infinity to finite coordinate values by defining,

$$\tilde{U} = \arctan\left(\frac{U}{\sqrt{2GM}}\right), \quad \tilde{V} = \arctan\left(\frac{V}{\sqrt{2GM}}\right), \tag{6.122}$$

with ranges

$$-\frac{\pi}{2} < \tilde{U} < \frac{\pi}{2}, \quad -\frac{\pi}{2} < \tilde{V} < \frac{\pi}{2}, \quad -\frac{\pi}{2} < \tilde{U} + \tilde{V} < \frac{\pi}{2}. \tag{6.123}$$

In these coordinates, the metric in (6.111) becomes

$$ds^2 = -\frac{64\, G^4 M^4}{r}\, e^{-r/(2GM)} \frac{1}{\cos^2 \tilde{U}\, \cos^2 \tilde{V}}\, d\tilde{U}\, d\tilde{V} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{6.124}$$

In these coordinates, the $(\tilde{U}, \tilde{V})$ part of the metric is conformally related to Minkowski space. In the new coordinates, the singularities at $r = 0$ are straight lines that stretch from timelike infinity in one region to timelike infinity in the other.

The conformal diagram of the maximally extended Schwarzschild solution is shown in Fig. 6.7. The only real subtlety about this diagram is the necessity to understand that i^- and i^+ are distinct from $r = 0$, since there are plenty of timelike curves that do not hit the singularity. As in the Kruskal diagram, light cones in the conformal

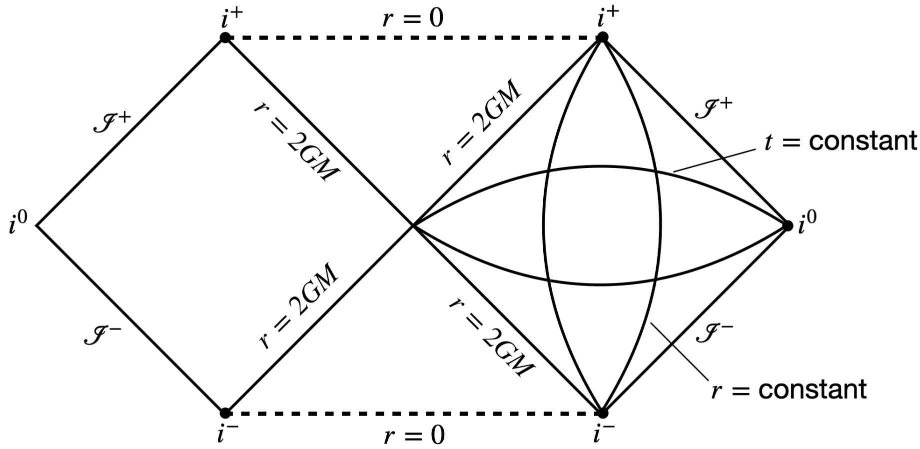


Figure 6.7: Penrose diagram for the Schwarzschild spacetime.

diagram are straight lines at 45°; the major difference is that the entire spacetime is now represented in a finite region. Notice that the structure of conformal infinity is just like that of Minkowski space, consistent with the fact that the Schwarzschild spacetime is asymptotically flat.

6.8 More general black holes

In this section we will discuss the properties of more general black holes. Most astrophysical objects such as stars, galaxies, etc., rotate and hence, if black holes form in natural astrophysical processes one would expect that they also rotate. As have seen in the previous sections, the Schwzarschild solution can describe a static and spherically symmetric black hole and hence, from the astrophysical point of view, not generic. Whilst the Schwzarschild solution was found weeks after Einstein published his theory of general relativity, an analytic solution of the Einstein equations describing an equilibrium rotating black hole was not found until 1963 by Kerr. The techniques needed to derive this spacetime are beyond the scope of this course; here we will simply give the Kerr metric:

$$\begin{aligned} ds^2 =& -\left(1 - \frac{2GMr}{\Sigma}\right) dt^2 - \frac{4GM\,a\,r\sin^2\theta}{\Sigma}\,dt\,d\phi + \frac{\Sigma}{\Delta}\,dr^2 \\ &+ \Sigma\,d\theta^2 + \frac{\sin^2\theta}{\Sigma}\left[(r^2+a^2)^2 - a^2\Delta\,\sin^2\theta\right] d\phi^2 \end{aligned} \tag{6.125}$$

where $\Delta = r^2 - 2GMr + a^2$ and $\Sigma = r^2 + a^2\cos^2\theta$. In (6.125) M corresponds to the mass of the spacetime and a is the angular momentum per unit mass,

$$a = J/M\,.$$

The coordinates (t, r, θ, ϕ) are known as Boyer-Lindquist coordinates and it is straightforward to see that, in these coordinates, $a \to 0$ reduces to the Schwarzschild spacetime. Note that if we keep a fixed and send $M \to 0$ we recover flat space in the so called ellipsoidal coordinates. These coordinates are related to the usual Cartesian coordinates by

$$\begin{aligned} x &= (r^2+a^2)^{\frac{1}{2}} \sin\theta \, \cos\phi \,, \\ y &= (r^2+a^2)^{\frac{1}{2}} \sin\theta \, \sin\phi \,, \\ z &= r \, \cos\theta \end{aligned} \tag{6.126}$$

One can show that the event horizons of the Kerr metric occur when

$$\Delta = r^2 - 2GMr + a^2 = 0 \,. \tag{6.127}$$

There are three possibilities: $GM > a$, $GM = a$ and $GM < a$. The last case corresponds to a naked singularity, while $GM = a$ is the extremal case, which is believed to be unstable. Both of these cases are of less physical interest and hence we will concentrate on the $GM > a$ case. Then, the function Δ vanishes at two values of the radial coordinate r, giving:

$$r_\pm = GM \pm \sqrt{(GM)^2 - a^2} \,. \tag{6.128}$$

Both radii are null surfaces corresponding to the inner ($r = r_-$) and outer ($r = r_+$) event horizons. Furthermore, one can show that the Kerr spacetime (6.125) has a physical curvature singularity at $\Sigma = r^2 + a^2 \cos^2\theta = 0$. Since this is the sum of two manifestly non-negative quantities, it can only vanish when both quantities are zero:

$$r = 0 \,, \quad \theta = \frac{\pi}{2} \,.$$

In these coordinates, $r = 0$ is not a point but a disc, see (6.126). Therefore, the set of points $r = 0$, $\theta = \frac{\pi}{2}$ is the ring at the edge of this disc. The Penrose diagram for the subextremal (i.e., $GM > a$) Kerr black hole is shown in Fig. 6.8.

The Kerr metric (6.125) is manifestly independent of the coordinates t and ϕ, and hence it possess two Killing vector fields, namely $K = \partial_t$ and $R = \partial_\phi$. Note however that K^a is not orthogonal to the $t =$ constant hypersurfaces. This metric is stationary but not static, which makes perfect sense: the black hole is rotating, so it is not static, but it is spinning in the same way at all times, so it is stationary. Alternatively, the metric cannot be static because it is not invariant under time reversals; to leave the metric invariant when going back in time $t \to -t$, one also has to reverse the direction of the rotation $\phi \to -\phi$.

Because the Kerr metric is stationary but not static, the event horizons $r_\pm$ are not Killing horizons of the asymptotic time-translation Killing vector $K = \partial_t$. The norm of K^a is given by

$$K^a \, K_a = -\frac{1}{\Sigma}(\Delta - a^2 \sin^2\theta) \,. \tag{6.129}$$

Note that at the outer horizon, $r = r_+$ (where $\Delta = 0$), we have $K^a K_a \geq 0$ so K^a is already spacelike there, except at the poles ($\theta = 0, \pi$), where it is null. The points where $K^a K_a = 0$ is a stationary limit surface known as the ergosphere and it is outside the outer event horizon. The region between the ergosphere and the outer event horizon is known as the ergoregion. Inside the ergoregion, observers must move in direction of rotation of the black hole (ϕ direction). This effect is known as frame dragging. However, observers can still move towards or away from the event horizon, and can exit the ergoregion.

The closest analog to a family of static observers outside the black hole in the Kerr geometry are the “locally non-rotating observers” (see Coursework 11) whose 4-velocity is given by $u^a = -\nabla^a t/(-\nabla_b t \nabla^b t)^{1/2}$. These observers rotate with coordinate angular velocity

$$\Omega = \frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{a(r^2 + a^2 - \Delta)}{(r^2+a^2)^2 - \Delta \, a^2 \, \sin^2\theta} \,. \tag{6.130}$$

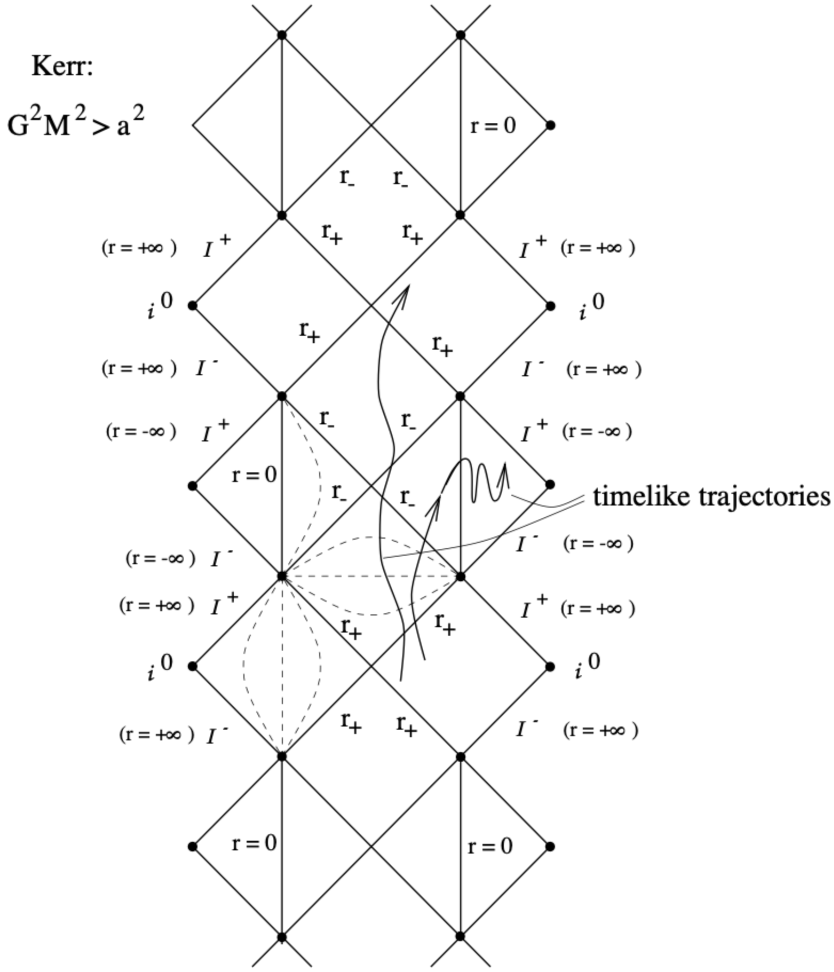


Figure 6.8: Penrose diagram of the subextremal Kerr black hole.

In the limit as one approaches the black hole outer event horizon, $r \to r_+$, this coordinate angular velocity becomes

$$\Omega_H = \frac{a}{r_+^2 + a^2}\,. \tag{6.131}$$

This is related to the fact that it is the Killing vector field

$$\chi = \frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \phi} \tag{6.132}$$

(rather than ∂_t) which is tangent to the null geodesic generators of the horizon of the Kerr black hole. Equation (6.132) can be interpreted as saying that the event horizon of the Kerr black hole rotates with angular velocity Ω_H.

Killing horizons

In the Schwarzschild metric, the Killing vector $K = \partial_t$ goes from being timelike to null at the event horizon. In general, if a Killing vector field χ^a is null along some hypersurface Σ, we say that Σ is a Killing horizon of χ^a. Note that the vector field χ^a will be normal to Σ, since a null surface cannot have two linearly independent null tangent vectors.

The notion of a Killing horizon is independent from that of an event horizon, but in spacetimes with time-translation symmetry the two are closely related. Under certain reasonable conditions, we have the following classification:

- Every event horizon $\mathcal{H}$ in a stationary, asymptotically flat spacetime is a Killing horizon for some Killing vector field χ^a.
- If the spacetime is static, $\chi^a = K^a = (\partial_t)^a$, the time-translations Killing vector field at infinity.
- If the spacetime is stationary but not static, then it must be axisymmetric with a rotational Killing vector field $R^a = (\partial_\phi)^a$, and χ^a will be a linear combination $K^a + \Omega_H \, R^a$ for some constant Ω_H.

To every Killing horizon we can associate a quantity called the surface gravity. Consider a Killing vector χ^a with Killing horizon Σ. Because χ^a is a normal vector to Σ, along the Killing horizon it obeys the geodesic equation,

$$\chi^a \nabla_a \chi^b = \kappa \, \chi^b \,, \tag{6.133}$$

where the right-hand side arises because the integral curves of χ^a may not be affinely parametrized. The parameter κ is the surface gravity and it will be constant on the horizon.

Uniqueness theorems and astrophysical relevance of the Kerr solution

Black holes can only be astrophysically relevant to describe certain compact objects in the Universe if they are stable. It has been shown that there are no unstable linear perturbations around the Kerr black hole. Furthermore, it is believed that the Kerr solution is dynamically stable at the full non-linear level and the evidence from the numerical simulations supports this expectation. This dynamical stability is a necessary condition for the Kerr black hole to be astrophysically relevant. In fact, the celebrated *Uniqueness Theorem* (Robinson,...) states that:

> Stationary, asymptotically flat black hole solutions to the Einstein vacuum equations are uniquely characterised by their mass and angular momentum and they are given by the Kerr family of solutions.

This theorem is also known as the *no-hair theorem* because it states that black holes are fully characterised by a small number of parameters, rather than potentially an infinite number of them as in the case of, for instance, a star. This is a very profound result since black holes are macroscopic objects and yet the theory of general relativity provides a complete mathematical description of them in terms of just a handful of parameters. Therefore, in some sense black holes are like elementary particles. Moreover, according this result, in general relativity *all* black holes in the Universe should be described by the Kerr solution.

Singularity theorems and cosmic censorship

Event horizons and black holes are important in general relativity because they are thought to be nearly inevitable. This conclusion follows from the celebrated singularity theorems and the cosmic censorship conjecture.

The ubiquity of singularities in general relativity is a consequence of the singularity theorems of Penrose (and Hawking and Penrose in the cosmological context).[9] Before these theorems were proven in the 60s, it was possible to hope that gravitational collapse to the Schwarzschild singularity was an accident of spherical symmetry and general spacetimes would be non-singular. The singularity theorems ensure that, under very general conditions, once gravitational collapse has reached a certain point, the formation of a singularity is inevitable. Therefore, according to general relativity, singularities should occur in Nature.

Singularities, however, can be problematic because general relativity breaks down near them and hence we cannot describe them within the theory itself. Therefore, the existence of singularities signals the incompleteness of the theory. The hope is that a would-be theory of quantum gravity resolves the singularities that appear in general relativity. It has been conjectured that even if singularities form generically according to the singularity theorems, they are always hidden behind event horizons. This is essentially the content of the cosmic censorship conjecture:

> Naked singularities cannot form in gravitational collapse from generic, initially non-singular states in an asymptotically flat spacetime with reasonable classical matter.

If this conjecture is true, then our ignorance regarding the physical description of singularities is irrelevant as far as the physics of the Universe outside black holes is concerned. Even though a general mathematical proof of the cosmic censorship conjecture is not available yet, there is very strong evidence that, in four spacetime dimensions (as in the presently observed Universe) this conjecture is true. However, in dimensions higher than four the cosmic censorship conjecture is false.

One consequence of the cosmic censorship conjecture is that classical black holes never shrink, they only grow bigger. Since the size of a black hole is measured by the area of the event horizon, we have Hawking's famous area theorem:

> Assuming the presence of reasonable classical matter[10] and cosmic censorship, the area of the future event horizon of a black hole in an asymptotically flat spacetime is non-decreasing.

Black hole thermodynamics

It was observed that, as a consequence of the Einstein equations, perturbing a black hole results in a change of its physical parameters given by

$$\delta M = \frac{\kappa}{8\pi\, G}\, \delta A + \Omega_H\, \delta J\,, \tag{6.134}$$

where A denotes the area of the event horizon. This equation, known as the first law of black hole mechanics, looks surprisingly similar to the first law of thermodynamics,

$$dE = T\, dS - p\, dV \tag{6.135}$$

[9]Thanks to the singularity theorems, Penrose was awarded the 2020 Nobel Prize of physics.

[10]By reasonable matter we mean matter that satisfies the so called weak energy condition: $\rho \geq 0$ and $\rho + p \geq 0$, where ρ is the energy density and p is any of the pressures.

where E is the energy, T the temperature, S the entropy, p is the pressure and V is the volume. Together with Hawking's area theorem, this suggested that there might exist an analogy between black holes and thermodynamics:

$$\begin{aligned} E &\leftrightarrow M \\ S &\leftrightarrow A/(4G) \\ T &\leftrightarrow \kappa/(2\pi) \end{aligned} \tag{6.136}$$

and the $\Omega_H \delta J$ term is the work that one does on the black hole when we throw matter into it. The second law of thermodynamics, which states that the entropy never decreases, is completely analogous to Hawking's area theorem with the above identification between the area of the event horizon and the entropy. At the classical level however, black holes do not have a temperature and their entropy would appear to be rather small since they are fully characterised in terms of a few parameters.

In equating $T\,dS$ to $\frac{\kappa}{8\pi G}\,dA$ we cheated a bit since we do not know how to separately normalise S/A or T/κ, only their combination. However, Hawking famously showed that when one considers quantum fields in the background of a black hole, the black hole evaporates emitting radiation at a temperature of $T = \kappa/(2\pi)$. This result shows that black holes do have a real temperature of $T = \kappa/(2\pi)$, and a real entropy given by $S = A/(4G)$; in astrophysical units, the entropy of a black hole is $S \sim 10^{90}\left(\frac{M}{M_\odot}\right)^2$, which is huge.[11] Restoring the units, these expressions become[12]

$$T = \frac{\hbar\,\kappa}{2\pi\,k_B}\,, \qquad S = \frac{k_B\,c^3}{4\,\hbar\,G}\,A\,, \tag{6.137}$$

where k_B is Boltzmann's constant. These expressions contain the fundamental constants of Nature, such as k_B, c, $\hbar$ and G, and they show that black holes not only link the fundamental laws of Nature, but they also provide a window into quantum gravity. In other words, any theory of quantum gravity must be able to reproduce this formula for the entropy of a black hole.

Given that black holes evaporate, their area decreases with time, which would violate the 2nd Law of Thermodynamics. To address this issue, Bekenstein proposed a generalised 2nd Law, which states that the combined entropy of matter and black holes never decreases:

$$\delta\left(S_{\text{matter}} + \frac{A}{4G}\right) \geq 0\,. \tag{6.138}$$

We have seen that the entropy of a black hole is huge, but from the point of statistical mechanics the entropy measures the number of accessible microscopic states compatible with a macroscopic equilibrium state. Given that a classical black hole is characterised by a small number of parameters (e.g., mass and angular momentum), it is hard to know what these states could be. Classically, this is not a big problem since any information about the microscopic state of a black hole could be hidden behind the horizon. Including quantum mechanics into this picture leads to the famous black hole information paradox. The reason is that, as Hawking showed, quantum mechanically black holes evaporate and hence, in a large but finite amount of time, there won't be any horizon left to hide the

[11] For matter fields in the Universe, the entropy is approximately equal to the number of relativistic particles; within the observable universe this number is $S_{\text{matter}} \sim 10^{88}$

[12] You can see this formula for the temperature in Stephen Hawking's tombstone at Westmister Abbey, where he lies next to Dirac and Newton.

microscopic states. Furthermore, Hawking argued that the radiation emitted by the black hole (known as Hawking radiation) is precisely thermal, and hence there should not be correlations between the emitted particles. Thus, all the information about the system before it collapsed into a black hole seems to have been erased by the time the black hole has evaporated. This is the black hole information loss paradox, and it suggests a potential incompatibility between quantum mechanics and general relativity.

Recent progress inspired by string theory (but independent of string theory) has shown that if one correctly accounts for the entanglement entropy of the Hawking particles emitted during the evaporation process, then there is no information loss. In fact, string theory is the only theory of quantum gravity that has been able to produce a complete microscopic description of certain black holes. By counting the microstates of the constituents of certain black holes in string theory, it has been possible to reproduce the Bekenstein-Hawking entropy of these black holes, including the subleading quantum corrections to the classical formula. This is regarded as one of the greatest triumphs of string theory. For other types of black holes, namely black holes in asymptotically anti-de Sitter spacetimes (not covered in the lectures), string theory provides a complete description in terms of a (non-gravitational) quantum field theory in one dimension less; because this quantum field theory obeys is standard laws of quantum mechanics, and hence its evolution is unitary, necessarily the evolution of the dual black hole, including the Hawking evaporation process, must also be unitary and therefore there cannot be information loss. The details of how this happens however, are still under investigation.

Chapter 7

Linearised theory and gravitational waves

7.1 Linearised theory

Solving the Einstein equation in general is tremendously difficult because it is a set of non-linear PDEs. However, in some circumstances the gravitational field is weak and the spacetime can be regarded as a perturbation of (flat) Minkowski space. In this context, we shall assume that the spacetime manifold is $\mathcal{M} = \mathbb{R}^4$ and that there exist globally defined "almost intertial" coordinates x^a for which the metric can be written as

$$g_{ab} = \eta_{ab} + h_{ab}\,, \quad \eta_{ab} = \mathrm{diag}\{-1,1,1,1\}\,, \tag{7.1}$$

where the components of h_{ab} are small compared to 1 to reflect the weakness of the gravitational field. Note that the spacetime metric is g_{ab}, so free particles move on geodesics of this metric. In the linearised theory, we regard h_{ab} as a tensor field in the sense of special relativity, i.e., it transforms as a tensor under Lorentz transformations.

To leading order in the perturbations around Minkowski space, the inverse of the spacetime metric is

$$g^{ab} = \eta^{ab} - h^{ab}\,, \tag{7.2}$$

where $h^{ab} = \eta^{ac}\eta^{bd}h_{cd}$. Therefore, from now on, we will raise and lower indices using the Minkowski metric η_{ab}. This agrees, to leading order, with raising and lowering indices with the full spacetime metric g_{ab}. Indeed, one can show that $g^{ac}g_{cb} = \delta^a{}_b + O(h^2)$.

Now we are going to write down Einstein's equation to first order in the metric perturbation h_{ab}. Recall that the structure of the Einstein equation is of the form,

$$G \sim \partial\Gamma + \partial\Gamma + \Gamma\,\Gamma + \Gamma\Gamma\,, \tag{7.3}$$

since the Einstein tensor is obtained from contractions of the Riemann tensor. Therefore, to leading order we only have to worry about the linear terms in the Christoffel symbols. To first order, the Christoffel symbols are

$$\Gamma^a{}_{bc} = \frac{1}{2}\,\eta^{ad}\left(\partial_b h_{cd} + \partial_c h_{bd} - \partial_d h_{bc}\right)\,, \tag{7.4}$$

and the Riemann tensor is,

$$\begin{aligned} R_{abcd} &= \eta_{ae}\left(\partial_c\Gamma^e{}_{bd} - \partial_d\Gamma^e{}_{bc}\right) \\ &= \frac{1}{2}\left(\partial_b\partial_c h_{ad} + \partial_a\partial_d h_{bc} - \partial_a\partial_c h_{bd} - \partial_b\partial_d h_{ac}\right)\,, \end{aligned} \tag{7.5}$$

and the linearised Ricci tensor is,

$$R_{ab} = \partial^c \partial_{(a} h_{b)c} - \frac{1}{2}\, \partial^c \partial_c h_{ab} - \frac{1}{2}\, \partial_a \partial_b h\,, \tag{7.6}$$

where $h = h^a{}_a = \eta^{ab} h_{ab}$. Finally, we obtain the linearised Einstein tensor:

$$G_{ab} = \partial^c \partial_{(a} h_{b)c} - \frac{1}{2}\, \partial^c \partial_c h_{ab} - \frac{1}{2}\, \partial_a \partial_b h - \frac{1}{2}\, \eta_{ab} \left(\partial^c \partial^d h_{cd} - \partial^c \partial_c h \right)\,. \tag{7.7}$$

The RHS of the Einstein equation is $8\pi\, G\, T_{ab}$ therefore, by consistency, we must assume that T_{ab} is also small. To proceed, it is convenient to define

$$\bar{h}_{ab} = h_{ab} - \frac{1}{2}\, h\, \eta_{ab}\,, \tag{7.8}$$

with the inverse $h_{ab} = \bar{h}_{ab} - \frac{1}{2}\, \bar{h}\, \eta_{ab}$ and $\bar{h} = \bar{h}^a{}_a = -h$. In terms of the new metric perturbation $\bar{h}_{ab}$, the linearised Einstein equation becomes

$$-\frac{1}{2}\, \partial^c \partial_c \bar{h}_{ab} + \partial^c \partial_{(a} \bar{h}_{b)c} - \frac{1}{2}\, \eta_{ab}\, \partial^c \partial^d \bar{h}_{cd} = 8\pi\, G\, T_{ab}\,. \tag{7.9}$$

Note that the first term in this equation looks like the typical wave operator acting on the the tensor field $\bar{h}_{ab}$. However, there are two extra terms that obscure the wave-like nature of the equation. As we shall see now, these terms can be removed using the freedom to choose coordinates in the theory. This freedom is also known as gauge freedom. Consider an infinitesimal coordinate transformation $x^a \to x^a + \xi^a$, where $\xi^a = \xi^a(x)$ is of the same order as h. Then, since the line element is invariant under coordinate transformations, we have

$$\begin{aligned}
ds^2 &= (\eta_{ab} + h_{ab})\, dx^a\, dx^b \\
&= (\eta_{ab} + h_{ab})\, d(x^a + \xi^a)\, d(x^b + \xi^b)\,, \\
&= (\eta_{ab} + h_{ab})\, dx^a\, dx^b + \eta_{ab}\, \partial_c \xi^a\, dx^c\, dx^b + \eta_{ab}\, \partial_c \xi^b\, dx^a\, dx^c \\
&= (\eta_{ab} + h_{ab} + \partial_a \xi_b + \partial_b \xi_a)\, dx^a\, dx^b + O(\epsilon^2)
\end{aligned} \tag{7.10}$$

Therefore, under an infinitesimal coordinate transformation, the metric perturbation transforms as

$$h_{ab} \to h_{ab} + \partial_a \xi_b + \partial_b \xi_a\,. \tag{7.11}$$

This implies that h_{ab} and $h_{ab} + \partial_a \xi_b + \partial_b \xi_a$ are physically equivalent. From the transformation of h_{ab} under infinitesimal coordinate transformations, it follows that

$$\bar{h}_{ab} \to \bar{h}_{ab} + \partial_a \xi_b + \partial_b \xi_a - \eta_{ab}\, \partial^c \xi_c\,. \tag{7.12}$$

We can use this symmetry to choose ξ^a to simplify the equations as much as possible. In particular, it is always possible, and convenient, to choose ξ^a such that $\bar{h}_{ab}$ satisfies,

$$\partial^a \bar{h}_{ab} = 0\,. \tag{7.13}$$

To see this, note that under (7.12) we have

$$\partial^a \bar{h}_{ab} \to \partial^a \bar{h}_{ab} + \partial^a \partial_a \xi_b\,. \tag{7.14}$$

So, if (7.13) is not obeyed, we can choose ξ_a to satisfy a wave equation $\partial^a \partial_a \xi_b = -\partial^a \bar{h}_{ab}$, which always has a solution. The gauge (7.13) is known as the Lorentz, de Donder or harmonic gauge, and in these coordinates the linearised Einstein equation becomes

$$\partial^c \partial_c \bar{h}_{ab} = -16\pi\, G\, T_{ab}\,. \tag{7.15}$$

This is a standard wave equation, which has a unique solution given appropriate boundary conditions.

7.2 Gravitational waves

The linearised Einstein equations in vacuum can be reduced to the following source-free wave equation for the metric perturbation $\bar{h}_{ab}$,

$$\partial^c \partial_c \bar{h}_{ab} = 0\,. \tag{7.16}$$

This shows that the theory admits wave solutions that propagate at the speed of light. As usual for the (linear) wave equation, we can construct any solution by superposing plane wave solutions. The latter are of the form,

$$\bar{h}_{ab}(x) = \mathrm{Re}\left(H_{ab}\, e^{i k_c x^c}\right)\,, \tag{7.17}$$

where H_{ab} is a constant symmetric complex matrix describing the polarisation of the wave and k^a is the wave vector. From now on, we shall suppress 'Re' from all the equations.

The wave equation (7.17) implies

$$k_a k^a = 0\,. \tag{7.18}$$

Therefore, the wave vector k^a must be null, in accordance with the fact that gravitational waves propagate at the speed of light. The gauge condition (7.13) implies

$$k^a H_{ab} = 0\,, \tag{7.19}$$

which shows that the waves are transverse (i.e., the polarisation vectors lie on the plane orthogonal to the direction of propagation).

The gauge condition (7.13) does not eliminate all gauge freedom. It follows from (7.14) that we can still perform a gauge transformation (7.14) that preserves the gauge condition (7.13) if ξ^a obeys a source-free wave equation,

$$\partial^c \partial_c \xi_a = 0\,. \tag{7.20}$$

We can use this residual gauge freedom to simplify the solution further. Consider,

$$\xi_a(x) = X_a\, e^{i k_c x^c}\,, \tag{7.21}$$

that satisfies (7.20) since k^a is null. Using

$$\bar{h}_{ab} \to \bar{h}_{ab} + \partial_a \xi_b + \partial_b \xi_a - \eta_{ab}\, \partial^c \xi_c\,, \tag{7.22}$$

under a gauge transformation, we see that the residual gauge freedom in our case is

$$H_{ab} \to H_{ab} + i\left(k_a\, X_b + k_b\, X_a - \eta_{ab}\, k^c\, X_c\right)\,. \tag{7.23}$$

This residual gauge freedom can be used to achieve a "longitudinal" gauge,

$$H_{0a} = 0\,. \tag{7.24}$$

This does not fully determine X_a. There is some remaining freedom that can be used to impose the additional "trace-free" condition,

$$H^a{}_a = 0\,. \tag{7.25}$$

In this "transverse-traceless" gauge we have

$$h_{ab} = \bar{h}_{ab} \,. \tag{7.26}$$

Example: Consider a wave travelling along the z-direction, $k^a = \omega(1,0,0,1)$. The longitudinal gauge condition (7.24) together with the transversality condition (7.19) implies $H_{3a} = 0$. Using the trace-free condition gives,

$$H_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_\times & 0 \\ 0 & H_\times & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{7.27}$$

This solution is specified by two constants, H_+ and $H_\times$, corresponding to the two independent polarisations of the gravitational waves. This shows that the gravitational field has two independent degrees of freedom per spacetime point.

7.2.1 Tidal accelerations and polarisation of gravitational waves

To measure the effect of gravitational waves on free particles we consider the geodesic deviation equation (5.8). Consider two nearby freely falling particles, which therefore move along two nearby geodesics. In addition, consider a local inertial frame at the point of the first geodesic, where the connecting vector ξ^a between the two geodesics originates. Let U^a denote the vector tangent to the geodesic. Hence, in this situation the geodesic deviation equation becomes

$$\nabla_U \nabla_U \xi^a = R^a{}_{bcd} U^b U^c \xi^d \,. \tag{7.28}$$

In this frame, coordinate distances are proper distances, as long as we can neglect quadratic terms in the coordinates. This means that in these coordinates the components of ξ^a correspond to measurable distances if the geodesics are near enough to one another. Furthermore, in this frame the second covariant derivative in (7.28) simplifies. The first derivative acting on ξ^a gives $\frac{d\xi^a}{d\tau}$, but the second one is a covariant one, which contains $\frac{d}{d\tau}$ but also a term with the Christoffel symbols. However, in a local inertial frame the Christoffel symbols all vanish at this point and therefore the second covariant derivative is just an ordinary derivative with respect to the proper time τ. Hence, in a locally inertial frame (7.28) becomes

$$\frac{d^2\xi^a}{d\tau^2} = R^a{}_{bcd} U^b U^c \xi^c \,, \tag{7.29}$$

where $U^a = \frac{dx^a}{d\tau}$ is the four-velocity of the two particles. In these coordinates the components of U^a are needed to the lowest order around flat space since any corrections to U^a will depend on h_{ab} and hence they will give rise to terms that are second order in h_{ab} in the equation above (because $R^a{}_{bcd}$ is already first order in h_{ab}). Therefore, under these approximations we can write, without loss of generality, $U^a = (1,0,0,0)$ and, initially, $\xi^a = (0,\varepsilon,0,0)$. Then, to first order in h_{ab}, equation (7.29) reduces to

$$\frac{d^2\xi^a}{d\tau^2} = \frac{\partial \xi^a}{\partial t^2} = \varepsilon\, R^a{}_{ttx} = -\varepsilon\, R^a{}_{txt} \,. \tag{7.30}$$

This equation shows that the Riemann tensor is locally measurable by simply watching the changes in the proper distance between nearby freely falling particles.

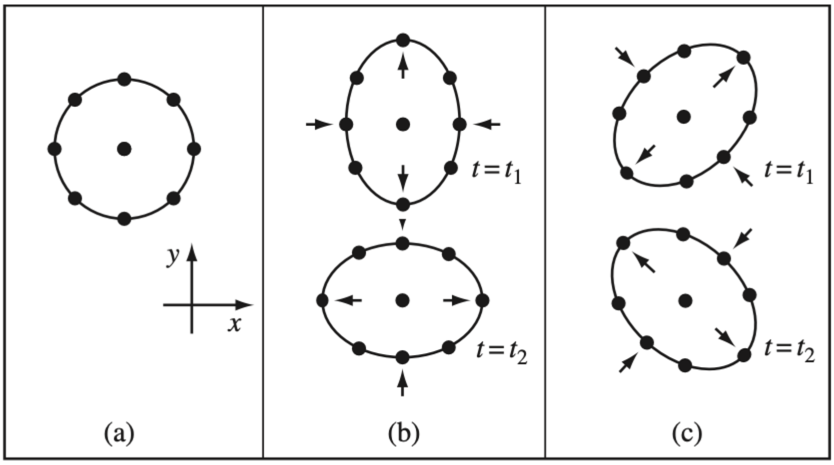


Figure 7.1: (a) Circle of free particles before a gravitational wave travelling in the z direction reaches them. (b) Distortions of the circle produced by a wave with the '+' polarisation. The two pictures show the same wave at phases separated by 180°. (c) As in (b) but now for the '×' polarisation.

The non-vanishing of the Riemann tensor is gauge invariant so the left-hand-side of (7.30) must have an interpretation that is independent of the coordinates. We identify ξ^a as the proper lengths of the components of the connecting vector, that is, the proper distances along the four coordinate directions over the coordinate intervals spanned by the vector.

We can now write (7.30) in the transverse traceless (TT) gauge. For a wave travelling in the z direction, the components of the Riemann tensor are

$$\begin{aligned} R^x{}_{txt} &= R_{xtxt} = -\frac{1}{2}\,\partial_t^2 h_{xx}^{\mathrm{TT}}\,, \\ R^y{}_{txt} &= R_{ytxt} = -\frac{1}{2}\,\partial_t^2 h_{xy}^{\mathrm{TT}}\,, \\ R^y{}_{tyt} &= R_{ytyt} = -\frac{1}{2}\,\partial_t^2 h_{yy}^{\mathrm{TT}} = -R^x{}_{txt}\,, \end{aligned} \tag{7.31}$$

and all the other independent components vanishing. This means that two particles initially separated by ε in the x direction have a separation vector ξ^a whose components' proper lengths obey

$$\frac{\partial^2 \xi^x}{\partial t^2} = \frac{1}{2}\,\varepsilon\,\partial_t^2 h_{xx}^{\mathrm{TT}}\,, \quad \frac{\partial^2 \xi^y}{\partial t^2} = \frac{1}{2}\,\varepsilon\,\partial_t^2 h_{xy}^{\mathrm{TT}}\,. \tag{7.32}$$

Similarly, two particles initially separated by ε in the y direction obey,

$$\frac{\partial^2 \xi^y}{\partial t^2} = \frac{1}{2}\,\varepsilon\,\partial_t^2 h_{yy}^{\mathrm{TT}} = -\frac{1}{2}\,\varepsilon\,\partial_t^2 h_{xx}^{\mathrm{TT}}\,, \quad \frac{\partial^2 \xi^x}{\partial t^2} = \frac{1}{2}\,\varepsilon\,\partial_t^2 h_{xy}^{\mathrm{TT}}\,, \tag{7.33}$$

since $h_{yy}^{\mathrm{TT}} = -h_{xx}^{\mathrm{TT}}$ from the tracelessness condition.

Equations (7.32) and (7.33) define the *polarisation* of a gravitational wave. Consider a ring of particles initially at rest in the x–y plane as in Fig. 7.1 (a). Suppose that at the time the wave reaches the particles it has $h_{xx}^{\mathrm{TT}} \neq 0$, $h_{xy}^{\mathrm{TT}} = 0$. Then the particles will be moved (in terms of the proper distance relative to the particle at the centre) in the way shown in Fig. 7.1 (b), as the wave oscillates and $h^{\rm TT}_{xx} = -h^{\rm TT}_{yy}$ changes sign. If, instead, the wave had $h^{\rm TT}_{xy} \neq 0$ and $h^{\rm TT}_{xx} = h^{\rm TT}_{yy} = 0$, then the particles would be distorted as in Fig. 7.1 (c). Since $h^{\rm TT}_{xx}$ and $h^{\rm TT}_{xy}$ are independent, (b) and (c) provide a pictorial representation for the two physically distinct linear polarisations of the gravitational waves. Notice that the two polarisations are simply rotated 45° relative to one another. This contrasts with the two polarisation states of an electromagnetic wave, which are at 90° to each other.

7.2.2 The far field

As we shall see, gravitational waves are produced by moving masses. To see this, lets return to the linearised Einstein equation with a source:

$$\partial^c \partial_c \bar{h}_{ab} = -16\pi\, G\, T_{ab}\,. \tag{7.34}$$

This is a standard wave equation, and hence we can write down the solution using the retarded Green's function:

$$\bar{h}_{ab}(t, \mathbf{x}) = 4\, G \int d^3x' \, \frac{T_{ab}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\,, \tag{7.35}$$

where $|\mathbf{x} - \mathbf{x}'|$ is calculated using the Euclidean metric. Assume that matter is confined to a compact region near the origin of size d. Then, far from the source we have $r \equiv |\mathbf{x}| \gg |\mathbf{x}'| \sim d$, and hence we can expand

$$|\mathbf{x} - \mathbf{x}'|^2 = r^2 - 2\,\mathbf{x}\cdot\mathbf{x}' + \mathbf{x}'^2 = r^2 \left(1 - \frac{2\,\hat{\mathbf{x}}\cdot\mathbf{x}'}{r} + O\left(\frac{d^2}{r^2}\right)\right)\,, \tag{7.36}$$

where $\hat{\mathbf{x}} \equiv \mathbf{x}/r$. Therefore,

$$|\mathbf{x} - \mathbf{x}'| = r - \hat{\mathbf{x}}\cdot\mathbf{x}' + O(d^2/r)\,, \tag{7.37}$$

$$T_{ab}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}') = T_{ab}(t', \mathbf{x}') + \hat{\mathbf{x}}\cdot\mathbf{x}'(\partial_0 T_{ab})(t', \mathbf{x}') + \ldots \tag{7.38}$$

where $t' = t - r$. Now let τ denote the times scale over which T_{ab} is varying, so $\partial_0 T_{ab} \sim T_{ab}/\tau$. For example, if the source is a binary black hole system, then τ would correspond to the orbital period. Then, the second term in (7.38) is of order $(d/\tau)T_{ab}$. Note that d is the time that it takes light to cross the region containing the source. Therefore, $d/\tau \ll 1$ if the matter is moving non-relativistically. For most systems, including black hole binaries, this is indeed the case for most of the time. Therefore, we will assume this from now on. This assumption implies that the second term is negligible compared to the first and hence

$$\bar{h}_{ij}(t, \mathbf{x}) \approx \frac{4\,G}{r} \int d^3x' \, T_{ij}(t', \mathbf{x}')\,, \quad t' = t - r\,. \tag{7.39}$$

Note that we only need to consider the spatial components of $\bar{h}_{ab}$; the other components can be obtained from the gauge condition (7.13), which gives

$$\partial_0 \bar{h}_{0i} = \partial_j \bar{h}_{ji}\,, \quad \partial_0 \bar{h}_{00} = \partial_i \bar{h}_{0i}\,. \tag{7.40}$$

So, given $\bar{h}_{ij}$, the first equation can be integrated to give $\bar{h}_{0i}$, and then the second can be integrated to get $\bar{h}_{00}$.

To obtain $\bar{h}_{ij}$ we have to evaluate the integral on the RHS of (7.39). This can be done as follows. Since matter is compactly supported by assumption, we can integrate by parts and discard surface terms. We can also use the fact that the energy-momentum tensor is conserved, $\partial_a T^{ab} = 0$. Then,

$$\begin{aligned}
\int d^3x\, T^{ij} &= \int d^3x \left[\partial_k(T^{ik}x^j) - (\partial_k T^{ik})x^j\right] \\
&= -\int d^3x\, (\partial_k T^{ik})x^j \qquad \text{drop surface term} \\
&= \int d^3x\, (\partial_0 T^{0i})x^j \qquad \text{use conservation law} \\
&= \partial_0 \int d^3x\, T^{0i}x^j
\end{aligned} \tag{7.41}$$

Note that the LHS of this equation is symmetric in ij, and hence we have to symmetrise the RHS too:

$$\begin{aligned}
\int d^3x\, T^{ij} &= \partial_0 \int d^3x\, T^{0(i}x^{j)} \\
&= \partial_0 \int d^3x \left[\frac{1}{2}\,\partial_k(T^{0k}\,x^i\,x^j) - \frac{1}{2}(\partial_k T^{0k})x^i\,x^j\right] \\
&= -\frac{1}{2}\,\partial_0 \int d^3x(\partial_k T^{0k})x^i\,x^j\,, \qquad \text{drop surface term} \\
&= \frac{1}{2}\,\partial_0 \int d^3x(\partial_0 T^{00})x^i\,x^j\,, \qquad \text{use conservation law} \\
&= \frac{1}{2}\,\partial_0\partial_0 \int d^3x\, T^{00}\,x^i\,x^j\,, \\
&= \frac{1}{2}\,\ddot{I}_{ij}(t)\,,
\end{aligned} \tag{7.42}$$

where

$$I_{ij}(t) = \int d^3x\, T_{00}(t,\mathbf{x})\,x^i\,x^j\,, \tag{7.43}$$

is the quadrupole moment tensor, also known as the second moment of the energy density (note that to leading order, $T_{00} = T^{00}$ and $T_{ij} = T^{ij}$). This object is a proper tensor in the Cartesian sense, i.e., it transforms in the usual way under rotations of the Cartesian coordinates x^i. Hence, we have

$$\bar{h}_{ij}(t-r,\mathbf{x}) \approx \frac{2\,G}{r}\,\ddot{I}_{ij}(t-r)\,. \tag{7.44}$$

This result is valid for $r \gg d$ and $\tau \gg d$, and it describes the propagation of a disturbance moving away from the source at the speed of light. If the source is undergoing an oscillatory motion, e.g., a binary black hole system, then $\bar{h}_{ij}$ will describe waves with the same period τ as the motion of the source.

To obtain the remaining components of the gravitational field, we go back to (7.40). The first equation gives,

$$\partial_0\bar{h}_{0i} \approx \partial_j\left(\frac{2\,G}{r}\,\ddot{I}_{ij}(t-r)\right)\,. \tag{7.45}$$

Integrating with respect to time and using that $\partial_i r = x_i/r = \hat{x}_i$, gives

$$\begin{aligned}\bar{h}_{0i} \approx \partial_j \left(\frac{2\,G}{r}\,\dot{I}_{ij}(t-r)\right) &= -\,\frac{2\,G\,\hat{x}_j}{r^2}\,\dot{I}_{ij}(t-r) - \frac{2\,G\,\hat{x}_j}{r}\,\ddot{I}_{ij}(t-r) \\ &\approx -\,\frac{2\,G\,\hat{x}_j}{r}\,\ddot{I}_{ij}(t-r)\,,\end{aligned} \tag{7.46}$$

where in the final line we have assumed that in the radiation zone $r \gg \tau$. This allows us to neglect the term proportional to $\dot{I}_{ij}$ because it is smaller than the term that we have kept by a factor of τ/r. In the radiation zone, space and time derivatives are of the same order of magnitude.

Similarly, integrating the second equation in (7.40) we get

$$\bar{h}_{00} \approx \partial_i \left(-\frac{2\,G\,\hat{x}_j}{r}\,\dot{I}_{ij}(t-r)\right) \approx \frac{2\,G\,\hat{x}_i\,\hat{x}_j}{r}\,\ddot{I}_{ij}(t-r)\,. \tag{7.47}$$

These expressions are not quite right because when integrating (7.40) we should have included an arbitrary time-independent term in $\bar{h}_{0i}$, which would lead to a term in $\bar{h}_{00}$ linear in time, as well as an arbitrary time-indepedent term in $\bar{h}_{00}$. The latter can be determined from (7.35), which to leading order gives

$$\bar{h}_{00} \approx \frac{4\,G\,E}{r}\,, \qquad E = \int d^3\mathbf{x}'\,T_{00}(t',\mathbf{x}')\,, \tag{7.48}$$

where E is the total energy of the matter. Note that the leading-order time-dependent piece (7.47) is smaller than the time-independent part by a factor of $d^2/d\tau^2$, so in order to get this term from (7.35) we would have to go to a higher order. Similarly, we find

$$\bar{h}_{0i} \approx -\frac{4\,G\,P_i}{r}\,, \qquad P_i = -\int d^3\mathbf{x}'\,T_{0i}(t',\mathbf{x}')\,, \tag{7.49}$$

where P_i is the total 3-momentum of the matter. Note that the term in $\bar{h}_{00}$ that is linear in t is proportional to P_i.

Remark. We will see shortly that gravitational waves carry away energy, so why is the total energy of matter constant? In fact, the total energy of matter is *not* constant, but to see this one has to go beyond the linearised theory.

A final simplification is possible: we are free to choose our almost inertial coordinates to correspond to the "centre of momentum" frame, i.e., $P_i = 0$. If we do this, then E is the total mass of the matter, which we denote by M. Then, in this frame we have

$$\bar{h}_{00}(t,\mathbf{x}) \approx \frac{4\,G\,M}{r} + \frac{2\,G\,\hat{x}_i\,\hat{x}_j}{r}\,\ddot{I}_{ij}(t-r)\,, \quad \bar{h}_{0i}(t,\mathbf{x}) \approx -\frac{2\,G\,\hat{x}_j}{r}\,\ddot{I}_{ij}(t-r)\,. \tag{7.50}$$

7.2.3 Compact binaries: the inspiral phase

We now want to apply the general result of the previous section to derive the (leading order) Gravitational Wave (GW) of two compact objects orbiting around each other. Binary systems of this type, involving black holes which are 5–100 times heaview than the sun, are the most common source of gravitational waves currently detected by the Ligo-Virgo-Kagra (LVK) collaboration. The first event recorded in 2015 and announced in 2016 in the following paper `https://arxiv.org/abs/1602.03837` represents a direct experimental evidence of the existence of both gravitational waves and of compact objects that are well described by the black hole solutions that we saw in the previous chapter.

Let us introduce the main quantities needed to describe quantitatively these GWs. First, the distance between the source and the observer r is much bigger than the size of the binary's orbit a_o (for a circular motion you can identify this a_o with the radius of the orbit). Thus we have $r \gg a$. This means that we detect the signal much later than it is emitted. So if the wave is emitted at $t = 0$, it is convenient to introduce the concept of "retarded time" u

$$u = t - r \tag{7.51}$$

and both t and r are very large, but u is a quantity of order one. Mathematically we are interested in evaluating explicitly Eq. (7.44)

$$\bar{h}_{ij}(u, \hat{\mathbf{n}}) \approx \lim_{r,t\to\infty} \frac{2\,G}{r}\, \ddot{I}_{ij}(t-r)\,, \qquad \frac{\mathbf{x}}{r} \equiv \hat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\,, \tag{7.52}$$

where $\hat{\mathbf{n}}$ and u are kept fixed as $t, r \to \infty$. As you can easily check $\hat{\mathbf{n}}$ is a unit vector and it simply the orientation of the obaserver with respect to the binary parametrised by the angles $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$.

The observers on planeth earth are very far from the compact binaries sourcing the GWs detected by LVK we will be working within the following assumptions:

- The orbital motion of the compact objects is slow, *i.e.* their velocity v is substantially smaller than the speed of light (think for instance at cases where $\frac{v}{c} \ll 1$).
- The size of the orbit is substantially bigger than the size of the compact objects R_i, *i.e.* $R_i \ll a_o$). If the binary is composed by two Schwarzschild black holes we can identify R_i with the Schwarzschild radius R_S of each black hole (see Eq. (6.31)).

For bound orbits these two assumptions are related (see below) and they are the basis of the Post-Newtonian (PN) approach we will adopt. The basic idea is to solve the mechanical problem describing the motion of the compact objects in the Newtonian theory and then add the relativistic corrections as perturbative terms which are small as they are weighted by powers of $\frac{v}{c}$. Here we will provide an explicit derivation of the leading PN contribution for the GW form and for the energy carried away by the GW.

We saw at the beginning of Section 7.2 that GWs contain two independent physical polarisations (H_+ and $H_\times$) which are defined in the plan orthogonal to the direction of propagation $\vec{n}$. In order to generalise Eq. (7.27) we introduce the unit vectors $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$

$$\begin{aligned}\mathbf{e}_\theta &= (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) \equiv \frac{\partial\hat{\mathbf{n}}}{\partial\theta}\,,\\ \mathbf{e}_\phi &= (-\sin\phi, \cos\phi, 0) \equiv \frac{1}{\sin\theta}\frac{\partial\hat{\mathbf{n}}}{\partial\phi}\,.\end{aligned} \tag{7.53}$$

It is straightforward to check that $\mathbf{e}_\theta \cdot \mathbf{e}_\phi = 0$ and $\mathbf{e}_\theta \cdot \hat{\mathbf{n}} = \mathbf{e}_\phi \cdot \hat{\mathbf{n}} = 0$ (the latter follows immediately from $\hat{\mathbf{n}}^2 = 1$ and Eq. (7.53)). Then we can define the "plus" and the "cross" polarisations of the GW as

$$\bar{h}_+ = \frac{1}{\sqrt{2}}(e^i_\theta e^j_\phi - e^j_\theta e^i_\phi)\,\bar{h}_{ij}(u,\hat{\mathbf{n}})\,, \quad \bar{h}_\times = \frac{1}{\sqrt{2}}(e^i_\theta e^j_\theta - e^i_\phi e^j_\phi)\,\bar{h}_{ij}(u,\hat{\mathbf{n}})\,. \tag{7.54}$$

It is useful to perform a Fourier transform of the GW above to write it in frequency space (since the detectors are sensitive only to a particular frequency window)

$$W_{+,\times}(\omega, \hat{\mathbf{n}}) = \int du\; e^{i\omega u}\bar{h}_{+,\times}(u, \hat{\mathbf{n}})\,. \tag{7.55}$$

The binary's orbit

Since we are assuming $a_o \gg R_S$, we can approximate the compact objects with point-like particles. For simplicity we will ignore the possible rotation of each object around its axis (i.e. the spin) and so we can characterise them just through their mass: m_1 and m_2. The Newtonian theory provides the leading order description of the motion of this system. Then the total kinetic and potential energies of the binary are

$$T = \frac{1}{2}m_1\left(\frac{d\mathbf{x}_1}{dt}\right)^2 + \frac{1}{2}m_2\left(\frac{d\mathbf{x}_2}{dt}\right)^2 \ , \quad U = -G\frac{m_1 m_2}{|\mathbf{x}_1 - \mathbf{x}_2|} \ , \tag{7.56}$$

where $\mathbf{x}_i(t)$ describes the trajectory of the object $i = 1, 2$ in some inertial Cartesian system. It is convenient to rewrite these results in terms of a new set of variables

$$\mathbf{X}_{cm} = \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{m_1 + m_2} \ , \quad \mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2 \ , \tag{7.57}$$

where $\mathbf{X}_{cm}$ describes the position of the centre of mass of the binary, while $\mathbf{x}$ is the relative distance between the two objects. It is straightforward to check (see `Tutorial 10`) that (7.56) can be rewritten in terms of the total mass $m = m_1 + m_2$, the reduced mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$ and the variables in (7.57)

$$T = \frac{1}{2}m\left(\frac{d\mathbf{X}_{cm}}{dt}\right)^2 + \frac{1}{2}\mu\left(\frac{d\mathbf{x}}{dt}\right)^2 \ , \quad U = -G\frac{m\mu}{|\mathbf{x}|} \ . \tag{7.58}$$

Thus the problem decouples in two separate parts: the first term describes the motion of the centre of mass and the rest takes the same form of a fictitious object of mass μ moving in a central potential U. Since the potential energy is independent of $\mathbf{X}_{cm}$, no force acts on the centre of mass which move with uniform velocity. From now we choose an inertial frame where $\mathbf{X}_{cm} = 0$, which for obvious reason goes under the name of "centre of mass frame".

The interesting part is the relative motion between the two constituents of the binary which is captured by $\mathbf{x}$. In the Newtonian approximation we can solve this problem starting from the Lagrangian

$$\mathcal{L} = T - U = \frac{1}{2}\mu\left(\frac{d\mathbf{x}}{dt}\right)^2 + G\frac{m\mu}{|\mathbf{x}|} \ . \tag{7.59}$$

Then the Euler-Lagrange equation reads

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^i}\right) - \frac{\partial\mathcal{L}}{\partial x^i} = 0 \ , \tag{7.60}$$

from which it is easy to see that the angular momentum $\mathbf{L} = \mathbf{x} \times \mu\dot{\mathbf{x}}$ is conserved

$$\frac{dL_i}{dt} = \frac{d}{dt}\left[\epsilon_{ijk}x_j\left(\mu\dot{x}_k\right)\right] = \mu\epsilon_{ijk}\left(\dot{x}_j\dot{x}_k + x_j\ddot{x}_k\right) = 0 \ . \tag{7.61}$$

The first term the latest round parenthesis vanishes because the Levi-Civita symbol ϵ_{ijk} is antisymmetric in the exchange of any pair of indices while $(\dot{x}_j\dot{x}_k$ is clearly symmetric in the exchange $i \leftrightarrow j$. In order to see that the second term in the parenthesis vanishes for the same reason, one needs to use Eq. (7.60) which allows to rewrite $\mu\ddot{x}_k$ in terms of $\frac{\partial\mathcal{L}}{dx_k}$ and then use (7.60) to show that this derivative is proportional to x_k.

The conservation of the angular momentum has the following important consequence: the motion takes place in a plane which is orthogonal to the constant vector $\mathbf{L}$ (since at any instant both the position $\mathbf{x}$ and the velocity $\dot{\mathbf{x}}$ are orthogonal to $\mathbf{L}$). Since the Lagrangian does not depend explicitly on time there is another physical quantity that does not change during the motion: the energy (called also Hamiltonian H)

$$E = \dot{x}_i \frac{\partial \mathcal{L}}{\partial \dot{x}^i} - \mathcal{L}(x_j, \dot{x}_j) = \frac{1}{2}\mu \left(\frac{d\mathbf{x}}{dt}\right)^2 - G\frac{m\mu}{|\mathbf{x}|} \; . \tag{7.62}$$

It is again straightforward to check that $\frac{dE}{dt} = 0$ by using (7.60). Since the motion takes place in a plane, which we can choose to be parametrised by x_1 and x_2, it is convenient to introduce polar coordinates

$$x = \rho \cos\phi \; , \quad y = \rho \sin\phi \; . \tag{7.63}$$

When expressed in these variables the energy and the angular momentum reads

$$|\mathbf{L}| \equiv L = \mu\rho^2\dot{\phi} \; , \quad E = \frac{1}{2}\mu\dot{\rho}^2 + \frac{1}{2}\mu\rho^2\dot{\phi}^2 - \frac{Gm\mu}{\rho} \; . \tag{7.64}$$

By writing the energy in terms of the angular momentum we have

$$E = \frac{1}{2}\mu\dot{\rho}^2 + \underbrace{\left(\frac{L^2}{2\mu\rho^2} - \frac{Gm\mu}{\rho}\right)}_{V_{\text{eff}}} \; , \tag{7.65}$$

where we collected in the round parenthesis the effective potential. It is interesting to compare this result with that obtained in the full relativistic setup for the motion of particle in the Schwarzschild metric, see Eq. (6.57): apart from a constant shift (and for having set $\mu \to 1$ in the relativistic analysis), the two results differ because of the extra term proportional to $1/r^3$ in (6.57). By isolating $\dot{r}$ and $\dot{\phi}$ in (7.64), we obtain the following equations

$$\begin{aligned} \frac{d\rho}{dt} &= \pm\sqrt{\frac{2}{\mu}\left(E + \frac{Gm\mu}{\rho} - \frac{L^2}{\mu^2\rho^2}\right)} \\ \frac{d\phi}{d\rho} &= \frac{d\phi}{dt}\frac{1}{\frac{d\rho}{dt}} = \pm\frac{\frac{L}{\mu\rho^2}}{\sqrt{\frac{2}{\mu}\left(E + \frac{Gm\mu}{\rho} - \frac{L^2}{\mu^2\rho^2}\right)}} \; , \end{aligned} \tag{7.66}$$

From the result above, we see that there are three cases

- $E > 0$. In this case the trajectory describes an open motion where the two objects get closed till they reach a minimal distance ρ_{min}, where $\dot{\rho}$ vanishes. E correspond to the initial/final kinetic energy (i.e. $\frac{1}{2}\mu\dot{\rho}^2$ evaluated at $\rho \to \pm\infty$).

- $E = 0$. Also in this case the trajectory describes an open motion, but the initial/final kinetic energy vanishes.

- $E < 0$. The trajectory is closed (i.e. the objects orbit around each other). In general there is a minimum ρ_{min} and a maximum ρ_{max} distance and in both points $\dot{\rho}$

vanishes. It is possible to integrate the equations in (7.66) in terms of trigonometric functions. Let us provide the result for the resulting trajectory in a parametric form

$$
\begin{aligned}
t &= \sqrt{\frac{a_o^3}{Gm}}\,(\chi - e\sin\chi) \ , \\
\rho &= a_o(1 - e\cos\chi) \ , \\
\phi &= 2\arctan\left[\sqrt{\frac{1+e}{1-e}}\tan\frac{\chi}{2}\right] \ ,
\end{aligned}
\tag{7.67}
$$

with $a_o = \frac{Gm\mu}{2|E|}$ and $L = a_o\sqrt{2\mu|E|(1-e^2)}$. With some patience it is straightforward to check that (7.67) satisfies the equations (7.66).

We will focus on the case of the bound trajectories. Eqs. (7.67) describes an ellipse with major semi-axis a_o and eccentricity e. In the case $e = 0$, the ellipse degenerates into a circle. One can check this also by using (7.65) and imposing that the discriminant of the quadratic equation (for $\dot{\rho} = 0$ since in a circular motion the radius does not change) vanishes.

The GW for circular binaries

Consider a binary system consisting of two black holes of masses m_1 and m_2 moving in a circular orbit of radius a_o around each other on the (x, y)-plane, see Fig. 7.2. By using (7.67) with $e = 0$, one can see that $\phi = \chi$ and so, in this case, it is easy to parametrise the curve in terms t

$$
\rho = a_o \ , \quad \phi = \sqrt{\frac{Gm}{a_o^3}}t \ . \tag{7.68}
$$

By using (7.57), one can write the position of the black holes as a function of time is

$$
\begin{aligned}
\mathbf{x}_1 &= a_o\,\frac{\mu}{m_1}\,(\cos(\Omega t), \sin(\Omega t), 0) \ , \\
\mathbf{x}_2 &= a_o\,\frac{\mu}{m_2}\,(-\cos(\Omega t), -\sin(\Omega t), 0) \ ,
\end{aligned}
\tag{7.69}
$$

where

$$
\Omega = \sqrt{\frac{Gm}{a_o^3}} \tag{7.70}
$$

is the frequency of the orbit. Notice that this is Kepler's law for the case of circular orbits!

Since we are treating the black holes as point particles, we can straightforwardly write down the mass density in terms of delta functions that localise it onto the black holes

$$
\rho = \left[m_1\,\delta(x - x_1)\delta(y - y_1) + m_2\,\delta(x - x_2)\delta(y - y_2)\right]\delta(z) \ , \tag{7.71}
$$

where we can use (7.63). Now we can easily evaluate the components of the quadrupole moment tensor (7.43) since we can use the delta functions to carry out the integrals

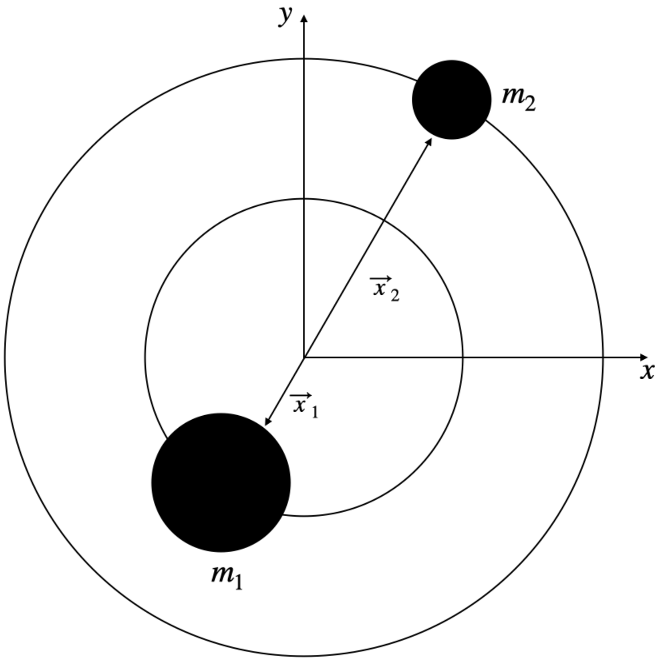


Figure 7.2: Unequal mass black hole binary.

obtaining

$$\begin{aligned} I_{xx} &= \int d^3x\, T_{00}\, x^2 \\ &= \int d^3x\, \rho\, x^2 = m_1\, x_1^2 + m_2\, x_2^2\,, \\ &= \mu^2\, a_o^2 \left(\frac{1}{m_1} + \frac{1}{m_2}\right) \cos^2(\Omega\, t)\,, \\ &= \frac{\mu\, a_o^2}{2}\, (1 + \cos(2\,\Omega\, t))\;. \end{aligned} \tag{7.72}$$

Note that this result shows that the frequency $\Omega_{\rm GW}$ of the gravitational wave is twice the orbital frequency: $\Omega_{\rm GW} = 2\Omega$. In other words, for each cycle made by the binary motion, the gravitational wave goes through two cycles and hence there are two maxima and two minima per orbit. For this reason, gravitational waves are called quadrupolar waves. The other non-vanishing components of the quadrupole moment tensor are

$$I_{yy} = \frac{\mu\, a_o^2}{2}\, (1 - \cos(2\,\Omega\, t))\;, \quad I_{xy} = \frac{\mu\, a_o^2}{2}\, \sin(2\,\Omega\, t)\,, \tag{7.73}$$

and the trace is

$$I_{ii} = \mu\, a_o^2\,. \tag{7.74}$$

Therefore, we can now write down the traceless part of the full quadrupole moment tensor:

$$Q_{ij} = I_{ij} - \frac{1}{3} I_{kk}\, \delta_{ij} = \frac{\mu\, a_o^2}{2} \begin{pmatrix} \cos(2\,\Omega\, t) + \frac{1}{3} & \sin(2\,\Omega\, t) & 0 \\ \sin(2\,\Omega\, t) & -\cos(2\,\Omega\, t) + \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}\,. \tag{7.75}$$

By using this result in (7.44), we obtain

$$\bar{h}_{ij}(u,\theta,\phi) = \frac{2G}{r}\left[-(2\mu a_o^2\Omega^2)\begin{pmatrix} \cos(2\,\Omega\,u) & \sin(2\,\Omega\,u) & 0 \\ \sin(2\,\Omega\,u) & -\cos(2\,\Omega\,u) & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] \equiv \frac{4G}{r}\left(\frac{1}{2}U_{ij}\right) . \tag{7.76}$$

Here U_{ij} represents the quadrupolar contribution to the GW and at subleading orders in the Post-Newtonian expansion, there will be further terms that are subleading in the $\frac{v}{c}$. By using (7.54), it is immediate to find the two leading order PN contribution to the physical polarisations

$$\begin{aligned} \bar{h}_\times &= \frac{2G}{r}\big[-\sqrt{2}(2\mu a_o^2\Omega^2)\cos\theta\sin(2\phi-2\Omega u)\big] , \\ \bar{h}_+ &= \frac{2G}{r}\left[(2\mu a_o^2\Omega^2)\frac{1+\cos^2\theta}{\sqrt{2}}\cos(2\phi-2\Omega u)\right] . \end{aligned} \tag{7.77}$$

Thus we obtain a monochromatic GW, but as we will see in the next section, this is just a artifact of the fact that we have not yet taken into account the energy carried away by the GW. This will change the picture introducing a subleading PN effect which modulates the wave.

7.2.4 Energy in gravitational waves

We have seen that gravitational waves arise whenever there is a non-trivial quadrupole moment I_{ij} that varies in time. To calculate the energy that gravitational waves carry we have to go to second order in perturbation theory:

$$g_{ab} = \eta_{ab} + h_{ab} + h^{(2)}_{ab} . \tag{7.78}$$

The result of the derivation given below can be summaries in terms of the physical polarisations as follows

$$P = \frac{1}{32\pi G}\sum_{a=+,\times}\int dt\int_0^\pi d\theta\sin\theta\int_0^{2\pi}d\phi\left(\frac{\partial(r\bar{h}_a)}{\partial t}\right)^2 . \tag{7.79}$$

If the components of h_{ab} are $O(\epsilon)$, then the components of $h^{(2)}_{ab}$ are $O(\epsilon^2)$. Now we have to calculate the Einstein tensor to second order. We have calculated the first order term in (7.7); lets call this piece $G^{(1)}_{ab}[h]$. The second order piece will contain terms that are linear in the second order perturbation, $h^{(2)}$, and terms that are quadratic in the first order perturbation, h. The terms linear in $h^{(2)}$ are simply given by (7.7) replacing $h \to h^{(2)}$; we denote them by $G^{(1)}_{ab}[h^{(2)}]$. Therefore, to second order we have

$$G_{ab}[g] = G^{(1)}_{ab}[h] + G^{(1)}_{ab}[h^{(2)}] + G^{(2)}_{ab}[h] , \tag{7.80}$$

where $G^{(2)}_{ab}[h]$ is the term in G_{ab} that is quadratic in h. This is given by

$$G^{(2)}_{ab}[h] = R^{(2)}_{ab}[h] - \frac{1}{2}R^{(1)}[h]\,h_{ab} - \frac{1}{2}R^{(2)}[h]\,\eta_{ab} , \tag{7.81}$$

where $R^{(2)}_{ab}[h]$ is the term in the Ricci tensor that is quadratic in h, and $R^{(1)}$ and $R^{(2)}$ are the terms in the Ricci scalar that are linear and quadratic in h respectively. We can write the latter as

$$R^{(2)}[h] = \eta^{ab}R^{(2)}_{ab}[h] - h^{ab}R^{(1)}_{ab}[h] . \tag{7.82}$$

A rather lengthy calculation gives,

$$\begin{aligned}R^{(2)}_{ab}[h] = \frac{1}{2}\, h^{cd}\partial_a\partial_b h_{cd} - h^{cd}\partial_c\partial_{(a}h_{b)d} + \frac{1}{4}\,(\partial_a h_{cd})(\partial_b h^{cd}) + (\partial^c h^d{}_b)\partial_{[c}h_{d]a} \\ + \frac{1}{2}\,\partial_c(h^{cd}\partial_d h_{ab}) - \frac{1}{4}\,(\partial^c h)(\partial_c h_{ab}) - \left(\partial_c h^{cd} - \frac{1}{2}\,\partial^d h\right)\partial_{(a}h_{b)d}\,.\end{aligned} \tag{7.83}$$

For simplicity, lets assume that no matter is present. Then, at first order the Einstein equation is $G^{(1)}_{ab}[h] = 0$. At second order, we have

$$G^{(1)}_{ab}[h^{(2)}] = 8\pi\, G\, t_{ab}[h]\,, \quad t_{ab}[h] \equiv -\frac{1}{8\pi\, G}\, G^{(2)}_{ab}[h]\,. \tag{7.84}$$

This is the equation of motion for $h^{(2)}$. Since h satisfies the linearised Einstein equation, we have $R^{(1)}_{ab}[h] = 0$ and hence

$$t_{ab}[h] = -\frac{1}{8\pi\, G}\left(R^{(2)}_{ab}[h] - \frac{1}{2}\,\eta^{cd}R^{(2)}_{cd}[h]\,\eta_{ab}\right)\,. \tag{7.85}$$

This object can be interpreted as the energy-momentum tensor of the gravitational field itself. As we shall show now, at this order in perturbation theory, it is conserved. To see this, consider the contracted Bianchi identity, $g^{bc}\nabla_c G_{ba} = 0$, valid for any metric g. Expanding this to first order we get,

$$\partial^a G^{(1)}_{ab}[h] = 0\,, \tag{7.86}$$

for an *arbitrary* first order perturbation h. At second order, one finds

$$\partial^a\left(G^{(1)}_{ab}[h^{(2)}] + G^{(2)}_{ab}[h]\right) + h\, G^{(1)}[h] = 0\,, \tag{7.87}$$

where the last term denotes schematically the terms that arise from a first order perturbation in the inverse metric and the Christoffel symbols. Now, since (7.86) holds for an arbitrary h, it also holds if we replace h by $h^{(2)}$ and hence $\partial^a G^{(1)}_{ab}[h^{(2)}] = 0$. Furthermore, if we assume that h satisfies the first order equation of motion then $G^{(1)}_{ab}[h] = 0$ and the last term (7.87) vanishes. Therefore, this equation reduces to

$$\partial^a t_{ab} = 0\,. \tag{7.88}$$

Hence, t_{ab} is a symmetric tensor that is (i) quadratic in the metric perturbation h, (ii) conserved if h satisfies the linear equation of motion, and (iii) appears on the RHS of the second order Einstein equation (7.84). Therefore, it seems rather natural to interpret this object as the energy-momentum tensor of the gravitational field, as we claimed earlier. However, there is a problem: t_{ab} is not invariant under a gauge transformation (7.11). This is how the impossibility of localising gravitational energy arises in the linearised theory, which is ultimately related to the fact that any spacetime is locally flat. Nevertheless it can be shown that the *integral* of t_{00} over a surface of constant time t is gauge invariant provided that one considers metric perturbations h_{ab} that decay sufficiently fast at infinity and restricts to gauge transformations that preserve this property. This integral provides a satisfactory notion of *total* energy in the linearised gravitational field. Therefore, gravitational energy exists but it is non-local.

One can use the second order Einstein equation (7.84) to convert the integral defining the energy, which is quadratic in h, into a surface integral at infinity which is linear in $h^{(2)}$.

In fact, the latter can be made fully non-linear in the sense that these integrals are valid in any asymptotically flat space, irrespective of whether the linearised approximation holds in the interior of the spacetime. This notion of energy is known as *ADM energy*.

We can proceed by following a more intuitive route and convert t_{ab} into a gauge-invariant quantity by averaging. This can be done as follows. For any point p, consider some region R of $\mathbb{R}^4$ of typical coordinate size ℓ centred on p. Define the average of a tensor X_{ab} at p by

$$\langle X_{ab} \rangle = \int_R d^4x \, W(x) \, X_{ab}(x) \,, \tag{7.89}$$

where the averaging function $W(x)$ is positive, satisfies $\int_R d^4x \, W = 1$, and tends smoothly to zero on ∂R. Note that it makes sense to integrate X_{ab} because we are treating it as a tensor in Minkowski space, and we can add tensors at different points.

We are interested in averaging in a region far from the source, in which the gravitational radiation has some typical wavelength λ (and hence $\tau \sim \lambda$). Assume that the components of X_{ab} vary on a region of typical size x. Since the wavelength of the radiation is λ, $\partial_a X_{bc}$ will have components of typical size x/λ. But the average is

$$\langle \partial_a X_{bc} \rangle = - \int_R d^4x \, \partial_a W(x) \, X_{bc}(x) \,, \tag{7.90}$$

where we have integrated by parts and used that $W = 0$ on ∂R. Now, $\partial_a W$ has components of order W/ℓ, so the RHS has components of order x/ℓ. Hence, if we choose $\ell \gg \lambda$ then the averaging has the effect of reducing $\partial_a X_{bc}$ by a factor of $\lambda/\ell \ll 1$. So if we choose $\ell \gg \lambda$ then we can neglect total derivatives inside averages. This implies that we can freely integrate by parts inside averages:

$$\langle A \partial B \rangle = \langle \partial(AB) \rangle - \langle (\partial A) B \rangle \approx -\langle (\partial A) B \rangle \,, \tag{7.91}$$

because $\langle \partial(AB) \rangle$ is a factor λ/ℓ smaller than $\langle (\partial A) B \rangle$. From now on we assume $\ell \gg \lambda$.

Using the linearised Einstein equation one can show that, in vacuum,

$$\langle \eta^{ab} R^{(2)}_{ab}[h] \rangle = 0 \,, \tag{7.92}$$

and hence the second term in $t_{\mu\nu}[h]$ averages to zero. Using this result, one finds

$$\langle t_{ab} \rangle = \frac{1}{32\pi \, G} \langle (\partial_a \bar{h}_{cd}) \partial_b \bar{h}^{cd} - \frac{1}{2} (\partial_a \bar{h}) \partial_b \bar{h} - 2 \, (\partial_c \bar{h}^{cd}) \partial_{(a} \bar{h}_{b)d} \rangle \,. \tag{7.93}$$

Furthermore, one can show that $\langle t_{ab} \rangle$ is gauge invariant.

7.2.5 The quadrupole formula

Now we are ready to calculate the energy loss from a compact source due to the emission of gravitational waves. The averaged energy flux 3-vector is $-\langle t_{0i} \rangle$. Consider a large sphere of radius r far away from the source. The unit normal to such sphere (in a surface of constant t) is $\hat{x}_i$. Hence, the average total energy flux across this sphere, i.e., the average power radiated across the sphere is

$$\langle P \rangle = - \int d\Omega \, r^2 \, \langle t_{0i} \rangle \, \hat{x}_i \,, \tag{7.94}$$

where $d\Omega$ is the standard volume element on a unit round S^2.

We can now substitute the results in §7.2.2 into (7.93), remembering that those results were derived in harmonic gauge. We get

$$\begin{aligned}\langle t_{0i}\rangle &= \frac{1}{32\pi\,G}\left\langle (\partial_0\bar{h}_{cd})\partial_i\bar{h}^{cd} - \frac{1}{2}\,(\partial_0\bar{h})\partial_i\bar{h}\right\rangle \\ &= \frac{1}{32\pi\,G}\left\langle (\partial_0\bar{h}_{jk})\partial_i\bar{h}_{jk} - 2(\partial_0\bar{h}_{0j})\partial_i\bar{h}_{0j} + (\partial_0\bar{h}_{00})\partial_i\bar{h}_{00} - \frac{1}{2}(\partial_0\bar{h})\partial_i\bar{h}\right\rangle .\end{aligned} \tag{7.95}$$

Since $\bar{h}_{jk}(t,\mathbf{x}) = \frac{2G}{r}\ddot{I}_{jk}(t-r)$ we have

$$\partial_0\bar{h}_{jk} = \frac{2G}{r}\,\dddot{I}_{jk}(t-r)\,, \tag{7.96}$$

$$\partial_i\bar{h}_{jk} = \left(-\frac{2G}{r}\,\dddot{I}_{jk}(t-r) - \frac{2G}{r^2}\,\ddot{I}_{jk}(t-r)\right)\hat{x}_i\,. \tag{7.97}$$

The second term is smaller than the first by a factor of $\tau/r \ll 1$, and hence negligible for large enough r. Therefore,

$$-\frac{1}{32\pi\,G}\int d\Omega\, r^2\,\langle(\partial_0\bar{h}_{jk})\partial_i\bar{h}_{jk}\rangle\hat{x}_i = \frac{G}{2}\langle\,\dddot{I}_{ij}\,\dddot{I}_{ij}\rangle_{t-r}\,. \tag{7.98}$$

On the RHS, the average is a time average, take over an interval $a \gg \lambda \sim \tau$ centered on the retarded time $t-r$.

Next we have $\bar{h}_{0j} = -(2\,G\,\hat{x}_k/r)\ddot{I}_{jk}(t-r)$. Hence,

$$\partial_0\bar{h}_{0j} = -\frac{2\,G\,\hat{x}_k}{r}\,\dddot{I}_{jk}(t-r)\,,\quad \partial_i\bar{h}_{0j} \approx \frac{2\,G\,\hat{x}_k}{r}\,\dddot{I}_{jk}(t-r)\,\hat{x}_i\,, \tag{7.99}$$

where in the second expression we have used that $\tau/r \ll 1$ to neglect the terms that arise from differentiating $\hat{x}_k/r$. Hence,

$$-\frac{1}{32\pi\,G}\int d\Omega\, r^2\langle -2(\partial_0\bar{h}_{0j})\partial_i\bar{h}_{0j}\rangle\hat{x}_i = -\frac{G}{4\pi}\langle\,\dddot{I}_{jk}\,\dddot{I}_{jl}\rangle_{t-r}\int d\Omega\,\hat{x}_k\,\hat{x}_l\,. \tag{7.100}$$

Now recall that $\int d\Omega\,\hat{x}_k\,\hat{x}_l$ is isotropic (i.e., rotationally invariant) and hence it must be equal to $\kappa\,\delta_{kl}$ for some constant κ. Taking the trace fixes $\kappa = \frac{4\pi}{3}$. Hence, the RHS above is

$$-\frac{G}{3}\langle\,\dddot{I}_{ij}\,\dddot{I}_{ij}\rangle_{t-r}\,. \tag{7.101}$$

Next we use that $\bar{h}_{00} = \frac{4GM}{r} + \frac{2G\hat{x}_j\hat{x}_k}{r}\ddot{I}_{jk}(t-r)$ to obtain

$$\partial_0\bar{h}_{00} = \frac{2\,G\,\hat{x}_j\,\hat{x}_k}{r}\,\dddot{I}_{jk}(t-r)\,, \tag{7.102}$$

and

$$\partial_i\bar{h}_{00} \approx \left(-\frac{4GM}{r^2} - \frac{2\,G\,\hat{x}_j\,\hat{x}_k}{r}\,\dddot{I}_{jk}(t-r)\right)\hat{x}_i \approx -\frac{2\,G\,\hat{x}_j\,\hat{x}_k}{r}\,\dddot{I}_{jk}(t-r)\,\hat{x}_i\,, \tag{7.103}$$

where we have neglected terms arising from differentiating $\hat{x}_j\hat{x}_k/r$ in the first equality because in the radiation zone ($\tau/r \ll 1$) they are negligible with respect to the second term that we have kept. In the second equality we have neglected the first term in brackets because this leads to a term in the integral proportional to $\langle \dddot{I}_{jk} \rangle$, which is the average of a derivative and thus negligible. Hence we have

$$-\frac{1}{32\pi G} \int d\Omega\, r^2 \, \langle (\partial_0 \bar{h}_{00}) \partial_i \bar{h}_{00} \rangle \, \hat{x}_i = \frac{G}{8\pi} \langle \dddot{I}_{ij} \dddot{I}_{kl} \rangle_{t-r} \, X_{ijkl} \,, \tag{7.104}$$

where

$$X_{ijkl} = \int d\Omega \, \hat{x}_i \, \hat{x}_j \, \hat{x}_k \, \hat{x}_l \,, \tag{7.105}$$

is another isotropic integral which we will evaluate shortly.

Next we use $\bar{h} = \bar{h}_{jj} - \bar{h}_{00}$ and the above results to obtain

$$\partial_0 \bar{h} = \frac{2\,G}{r} \dddot{I}_{jj}(t-r) - \frac{2\,G\,\hat{x}_j\,\hat{x}_k}{r} \dddot{I}_{jk}(t-r) \,, \tag{7.106}$$

$$\partial_i \bar{h} = \left(-\frac{2\,G}{r} \dddot{I}_{jj}(t-r) + \frac{2\,G\,\hat{x}_j\,\hat{x}_k}{r} \dddot{I}_{jk}(t-r) \right) \hat{x}_i \,, \tag{7.107}$$

and hence

$$-\frac{1}{32\pi G} \int d\Omega\, r^2 \left\langle -\frac{1}{2} (\partial_0 \bar{h}) \partial_i \bar{h} \right\rangle \hat{x}_i = G \left\langle -\frac{}{4} \dddot{I}_{jj} \dddot{I}_{kk} + \frac{1}{6} \dddot{I}_{jj} \dddot{I}_{kk} - \frac{1}{16\pi} \dddot{I}_{ij} \dddot{I}_{kl} \, X_{ijkl} \right\rangle . \tag{7.108}$$

Putting everything together we have

$$\langle P \rangle_t = G \left\langle \frac{1}{6} \dddot{I}_{ij} \dddot{I}_{ij} - \frac{1}{12} \dddot{I}_{ii} \dddot{I}_{jj} + \frac{1}{16\pi} \dddot{I}_{ij} \dddot{I}_{kl} \, X_{ijkl} \right\rangle_{t-r} . \tag{7.109}$$

To evaluate X_{ijkl}, we use the fact that any isotropic Cartesian tensor must be a product of Kronecker's delta factors, δ_{ij}, and Levi-Civita totally anti-symmetric tensor factors, ϵ_{ijk}. Since X_{ijkl} has rank 4, it can only be a product of δ's, so it must be of the form $X_{ijkl} = \alpha \, \delta_{ij} \, \delta_{kl} + \beta \, \delta_{ik} \, \delta_{jl} + \gamma \, \delta_{il} \, \delta_{jk}$, for some constants α, β and γ. The symmetry of X_{ijkl} implies that $\alpha = \beta = \gamma$. Taking the trace on the ij and on the kl indices fixes $\alpha = \frac{4\pi}{15}$. Therefore, the final term above gives

$$\frac{1}{60} \langle \dddot{I}_{ii} \dddot{I}_{jj} + 2 \dddot{I}_{ij} \dddot{I}_{ij} \rangle \,, \tag{7.110}$$

and hence

$$\langle P \rangle_t = \frac{G}{5} \left\langle \dddot{I}_{ij} \dddot{I}_{ij} - \frac{1}{3} \dddot{I}_{ii} \dddot{I}_{jj} \right\rangle_{t-r} . \tag{7.111}$$

Finally, considering the traceless part of the mass/energy quadrupole moment tensor I_{ij},

$$Q_{ij} = I_{ij} - \frac{1}{3} I_{kk} \, \delta_{ij} \,. \tag{7.112}$$

we have

$$\langle P \rangle_t = \frac{G}{5} \langle \dddot{Q}_{ij} \dddot{Q}_{ij} \rangle_{t-r} \,. \tag{7.113}$$

This is the celebrated *quadrupole* formula for the power (i.e., energy loss) radiated via gravitational wave emission. It is valid in the radiation zone far from a non-relativistic source, i.e., for $r \gg \tau \gg d$.

We conclude that a body whose quadrupole tensor is varying in time will emit gravitational radiation. A spherically symmetric body has $Q_{ij} = 0$ and hence it cannot radiate. This is in agreement with *Birkhoff's theorem*, which asserts that the unique

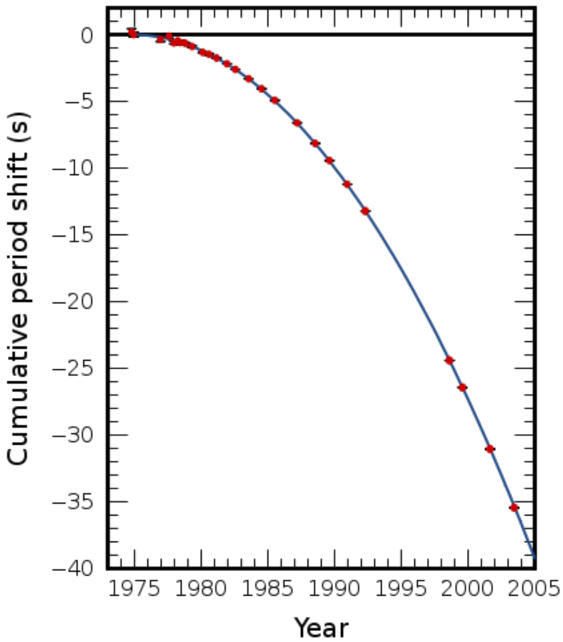


Figure 7.3: Decrease of the period of the famous Hulse-Taylor binary pulsar over the years due to the emission of gravitational waves. The solid curve is the GR prediction calculated using the quadrupole formula (7.113). This was the first (indirect) experimental evidence of the existence of gravitational waves.

spherically symmetric solution of the vacuum Einstein equation is the Schwarzschild solution. Hence, the spacetime outside a spherically symmetric body is time independent because it is described by the Schwarzschild solution.

Example: We can derive the energy loss for a circular binary system by using (7.79). We will restrict the integral over t to one period $T_{\rm orb} = \frac{2\pi}{\Omega}$ to calculate the energy radiated in one orbit $\langle P\rangle$. So by starting from (7.77) we have

$$\langle P\rangle = \frac{1}{32\pi G T_{\rm orb}} \sum_{a=+,\times} \int_0^{T_{\rm orb}} dt \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi \left(\frac{\partial (r\bar{h}_a)}{\partial t}\right)^2 . \tag{7.114}$$

Carrying out the t-integral by using

$$\int_0^{2\pi} \sin^2(2x+c)dx = \int_0^{2\pi} \cos^2(2x+c)dx = \frac{1}{2}$$

we obtain

$$\begin{aligned} \langle P\rangle &= \frac{1}{32\pi G} G^2\mu^2(4a_o^2\Omega^2)^2 4\Omega^2 \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi \left[\cos^2\theta + \frac{1+2\cos^2\theta+\cos^4\theta}{4}\right] \\ &= \frac{1}{32\pi G} G^2\mu^2 64 a_o^4\Omega^6 \frac{16\pi}{5} = \frac{32}{5}\frac{G^4 m_1^2 m_2^2 m}{a_o^5} = \frac{32\nu^2}{5G}\left(Gm\Omega\right)^{10/3} , \end{aligned} \tag{7.115}$$

where in the last steps we used Kepler's (7.70) law to write the result in terms of a_o or Ω and introduced $\nu = \frac{\mu}{m}$.

Exercise: For $m_1 = 36\,M_\odot$ and $m_2 = 29\,M_\odot$ ($M_\odot = 2 \times 10^{30}$kg is the mass of the Sun), and $R = 10^6$ light years, is $\langle P \rangle$ a large number?

The loss of energy results in the two black holes coming closer together, a process that is called inspiral. Using the virial theorem, the total energy of the binary is given by,

$$E = -\frac{G\,m_1\,m_2}{2a_o}\,. \tag{7.116}$$

Since the quadrupole formula (7.114) calculates the leading order rate of change of the energy of the binary, $\frac{dE}{dt} = \langle P \rangle$, using the chain rule, we can calculate the rate of change of the separation between the two black holes:

$$\frac{da_o}{dt} = \frac{da_o}{dE}\frac{dE}{dt} = -\left(\frac{2a_o^2}{Gm_1m_2}\right)\frac{32G^4}{5c^5}\frac{(m_1m_2)^2(m_1+m_2)}{a_o^5} = -\frac{64G^3}{5c^5}\frac{m_1m_2(m_1+m_2)}{a_o^3}\,. \tag{7.117}$$

Note that the negative sign indicates that the orbit is shrinking, so eventually the two black holes will merge. Given some initial separation $\bar{a}_0$, we can calculate the time to merger by integrating (7.117):

$$\Delta t_{\mathrm{merger}} = -\frac{5c^5}{64G^3m_1m_2(m_1+m_2)}\int_{\bar{a}_o}^{0} da_o\,a_o^3 = \frac{5c^5}{256G^3}\frac{(\bar{a}_o)^4}{m_1m_2(m_1+m_2)}\,. \tag{7.118}$$

Similarly we can calculate the rate of change of the orbital frequency. Looking at the last formulation of the radiated energy in one period (7.115) and writing (7.116) in terms of Ω, one can derive the following relation

$$\frac{1}{\Omega}\frac{d\Omega}{dt} = \frac{96}{5}\frac{\nu}{Gm}(Gm\Omega)^{8/3}\,. \tag{7.119}$$

Thus the variation of Ω in time is

$$d\Omega = (Gm\nu)^{5/3}\Omega^{11/3}dt \equiv (GM_{\mathrm{chirp}})^{5/3}\Omega^{11/3}dt\,, \tag{7.120}$$

which means that Ω increases and $M_{\mathrm{chirp}} \equiv \frac{(m_1m_2)^{3/5}}{m^{1/5}}$, called the chirp mass, is the parameter determining how quickly the "pitch" of the wave changes. It is one of the key parameters extracted from experimental data of each GW and gives a first information about the nature of the constituents of the binary providing a piece of information about their masses.