MTH5103 Complex Variables

Week 2 Practice Exercies

These exercises are for your daily practice.

1. Write down the definition for the convergence/absolute convergence/conditional convergence of a real infinite series. Can you give examples of each kind? Prove $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely for all $x \in \mathbb{R}$.

2. Verify the identity $\cos z + i \sin z = e^{iz} = \cosh(iz) + \sinh(iz)$.

3. Rewrite the following complex functions as real mappings: $\frac{3z}{\overline{z}} = 1, \Im(z^3) = 5, |z| = \Re z + \frac{1}{2}$.

4. What is the image of a horizontal line $y = k \in \mathbb{R}$ under the mapping $w = z^2$?

5. Prove that the mapping $f(z) = \frac{1}{z}$ satisfies the following:

   (a) The unit circle is mapped to itself.

   (b) In the first quadrant, the interior of unit circle maps to the exterior of unit circle in the fourth quadrant.

   (c) In the second quadrant, the exterior of unit circle maps to the interior of unit circle in the third quadrant.

   (d) Complete the remaining cases (as depicted in the diagram in lecture with regions $1, \ldots, 8$ mapping to regions $1', \ldots, 8'$.

6. Complete the squares appropriately to determine the radius of the circle given by the equation $a(x^2 + y^2) + bx + cy + d = 0$. Here, we assume $a \neq 0$.

7. Verify that the half plane above the horizontal line $y = k$ is mapped into the interior of the circle centred at $\left(0, \frac{-1}{2k}\right)$ of radius $\frac{1}{2k}$ under the mapping $z \mapsto \frac{1}{z}$.

8. Verify that the composition of two Möbius transformations is another Möbius transformation. Verify also that the inverse of the Möbius transformation having associated matrix M is given by M^{-1}, the inverse matrix.

9. (Harder) show that given any three distinct points z_1, z_2, z_3 in the z-plane, and any three points w_1, w_2, w_3 in the w-plane, there exists a Möbius transformation sending $z_j \mapsto w_j$ for each $j = 1, 2, 3$.