MTH6107 Chaos & Fractals

Solutions 7

Exercise 1. Let $F_k = \Phi^k([0,1]^2)$, where $\Phi(A) = \cup_{i=1}^5 \phi_i(A)$ for subsets $A \subset \mathbb{R}^2$, and where the five maps $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5$ are defined by $\phi_i(x,y) = (x_i, y_i) + (x/3, y/3)$, where $(x_1, y_1) = (0,0)$, $(x_2, y_2) = (2/3, 0)$, $(x_3, y_3) = (1/3, 1/3)$, $(x_4, y_4) = (0, 2/3)$, $(x_5, y_5) = (2/3, 2/3)$.

(a) Determine the set F_1.

(b) If F_k is expressed as a union of N_k closed squares, where the intersection of any two squares is either a singleton set or the empty set, compute the number N_k.

(c) What is the common side length of each of the N_k squares in (b) above?

(d) What is the box dimension of $F = \cap_{k=0}^\infty F_k$?

(a) The set F_1 is the union of 5 squares, each of side length $1/3$. More precisely, if $S = [0, 1/3] \times [0, 1/3]$, and we define $S_i = (x_i, y_i) + S$ for $1 \le i \le 5$, then $F_1 = \cup_{i=1}^5 S_i$. More explicitly, the squares $S_1, \ldots, S_5$ can be written as

$$S_1 = [0, 1/3] \times [0, 1/3], \ S_2 = [2/3, 1] \times [0, 1/3], \ S_3 = [1/3, 2/3] \times [1/3, 2/3],$$

$$S_4 = [0, 1/3] \times [2/3, 1], \ S_5 = [2/3, 1] \times [2/3, 1] .$$

(b) $N_k = 5^k$.
(c) The common side length is $1/3^k$.
(d) The box dimension is $\log 5 / \log 3$.

This can be seen by using the box dimension formula $D(F) = \log \beta / \log(1/\alpha)$, where α is the scaling factor for the side length of boxes, and β is the scaling factor for number of boxes.

Alternatively, if $\epsilon_k = 1/3^k$ denotes the common side length then we can compute that

$$D(F) = \lim_{k\to\infty} \frac{\log N_k}{\log(1/\epsilon_k)} = \lim_{k\to\infty} \frac{\log 5^k}{\log(1/(1/3^k))} = \lim_{k\to\infty} \frac{k \log 5}{k \log 3} = \lim_{k\to\infty} \frac{\log 5}{\log 3} = \frac{\log 5}{\log 3} .$$

Exercise 2. Let $C_0 = [0,1]$. In the standard construction of the middle third Cantor set $C = \cap_{k=0}^\infty C_k$, describe briefly how the sets C_k are defined, and explicitly write down the sets C_1 and C_2. If C_k is expressed as a disjoint union of N_k closed intervals, compute the number N_k, and determine the common length of each of the N_k closed intervals. Use this to show that, assuming the box dimension of the middle third Cantor set C exists, then it must equal $\log 2 / \log 3$

The set C_{k-1} is a disjoint union $\cup_i I_i$ of closed intervals. If from each of these closed intervals I_i we remove the 'open middle third', we are left with a pair of closed intervals I_i^- and I_i^+, each of length a third the length of I. The union $\cup_i(I_i^- \cup I_i^+)$ of these intervals is then defined to be the set C_k.

$C_1 = [0, 1/3] \cup [2/3, 1]$, and $C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$.

$N_k = 2^k$ because $N_0 = 1$ and the recursive procedure doubles the number of intervals at each step.

The length of each interval in C_k is $1/3^k$, because the length of the closed intervals decreases by a factor of 3 at each step, and the length of $C_0 = [0, 1]$ is 1.

If $\varepsilon_k = 1/3^k$ then $N(\varepsilon_k) = 2^k$, and so the box dimension equals

$$\lim_{k\to\infty} \frac{\log N(\varepsilon_k)}{-\log \varepsilon_k} = \lim_{k\to\infty} \frac{k \log 2}{k \log 3} = \frac{\log 2}{\log 3}\,.$$

Exercise 3. If C is the middle-third Cantor set, what is the box dimension of the planar subset $C \times [0, 1] = \{(x, y) : x \in C, y \in [0, 1]\}$?

The set $C \times [0, 1]$ has box dimension equal to $\frac{\log 2}{\log 3} + 1$ (note that this equals the sum of the box dimension of C and the box dimension of $[0, 1]$).

One way to see this is to note that $[0, 1] \times C$ is equal to $\cap_{k=0}^{\infty} \Phi^k([0, 1]^2)$, where $\Phi(A) = \cup_{i=1}^{6} \phi_i(A)$, and the maps ϕ_i all shrink lengths by a factor of $1/3$, so that the 6 images of the unit square are such that three of them are stacked on top of the interval $[0, 1/3]$ and the other three are stacked on top of the interval $[2/3, 1]$. More specifically, we can define $\phi_1(x, y) = (x/3, y/3)$, $\phi_2(x, y) = (x/3, y/3+1/3)$, $\phi_3(x, y) = (x/3, y/3 + 2/3)$, $\phi_4(x, y) = (x/3 + 2/3, y/3)$, $\phi_5(x, y) = (x/3 + 2/3, y/3 + 1/3)$, $\phi_6(x, y) = (x/3 + 2/3, y/3 + 2/3)$.

Since the side length scaling factor is $\alpha = 1/3$, and number of boxes scaling factor is $\beta = 6$, the box dimension is equal to

$$\frac{\log \beta}{\log(1/\alpha)} = \frac{\log 6}{\log 3} = \frac{\log 3 + \log 2}{\log 3} = 1 + \frac{\log 2}{\log 3}\,.$$

Exercise 4. Let ϕ_1, ϕ_2 be the iterated function system defined by the two maps $\phi_1(x) = x/10$ and $\phi_2(x) = (x+3)/10$, with $\Phi(A) := \phi_1(A) \cup \phi_2(A)$, and let C_k denote $\Phi^k([0, 1])$ for $k \geq 0$. Write down the sets C_1 and C_2. If C_k is expressed as a disjoint union of N_k closed intervals, compute the number N_k, and determine the common length of each of the N_k closed intervals whose disjoint union equals C_k. Show that if the box dimension of $C = \cap_{k=0}^{\infty} C_k$ exists then it must equal $\log 2/ \log 10$.

$C_1 = [0, 1/10] \cup [3/10, 4/10]$, and

$$C_2 = [0, 1/100] \cup [3/100, 4/100] \cup [3/10, 31/100] \cup [33/100, 34/100].$$

$N_k = 2^k$ because $N_0 = 1$ and the recursive procedure doubles the number of intervals at each step.

The length of each interval is $1/10^k$, because the length of the closed intervals decreases by a factor of 10 at each step, and the length of $C_0 = [0, 1]$ is 1.

If $\varepsilon_k = 1/10^k$ then $N(\varepsilon_k) = 2^k$, and so the box dimension equals

$$\lim_{k\to\infty} \frac{\log N(\varepsilon_k)}{-\log \varepsilon_k} = \lim_{k\to\infty} \frac{k \log 2}{k \log 10} = \frac{\log 2}{\log 10}\,.$$

Exercise 5. Let ϕ_1, ϕ_2 be the iterated function system defined by the two maps $\phi_1(x) = x/10$ and $\phi_2(x) = (x+9)/10$, with $\Phi(A) := \phi_1(A) \cup \phi_2(A)$, and let C_k denote $\Phi^k([0,1])$ for $k \geq 0$. If C_k is expressed as a disjoint union of N_k closed intervals, compute the number N_k, and determine the common length of each of the N_k closed intervals whose disjoint union equals C_k. Show that if the box dimension of $C = \cap_{k=0}^{\infty} C_k$ exists then it must equal $\log 2 / \log 10$.

$N_k = 2^k$ because $N_0 = 1$ and the recursive procedure doubles the number of intervals at each step.

The length of each interval is $1/10^k$, because the length of the closed intervals decreases by a factor of 10 at each step, and the length of $C_0 = [0,1]$ is 1.

If $\varepsilon_k = 1/10^k$ then $N(\varepsilon_k) = 2^k$, and so the box dimension equals

$$\lim_{k\to\infty} \frac{\log N(\varepsilon_k)}{-\log \varepsilon_k} = \lim_{k\to\infty} \frac{k \log 2}{k \log 10} = \frac{\log 2}{\log 10}.$$

Exercise 6. For C as in Exercise 5, give a description of the members of C in terms of the digits of their decimal expansion. If $f : C \to C$ is defined by $f(x) = 10x \pmod 1$ then find a point $x \in C$ which has minimal period 2 under f.

C consists of those numbers between 0 and 1 which have a decimal expansion whose digits all equal either 0 or 9.

There are two points of minimal period 2, namely

$$1/11 = 0.090909\ldots \quad \text{and} \quad 10/11 = 0.909090\ldots$$

Exercise 7. Describe the construction of the *Sierpinski triangle* P, and show that if the box dimension of P exists then it must equal $\log 3 / \log 2$

Begin with a solid equilateral triangle, then sub-divide it into 4 congruent equilateral triangles, then remove the central triangle, leaving 3 solid equilateral triangles. Repeat the above step with each of the remaining 3 triangles, and continue the process ad infinitum.

Assuming (without loss of generality) that the initial equilateral triangle has side length 1, we see that $N(1/2) = 3$, and more generally $N(1/2^k) = 3^k$, so existence of the box dimension D means that

$$D = \lim_{\varepsilon\to 0} \frac{\log N(\varepsilon)}{-\log \varepsilon} = \lim_{k\to\infty} \frac{\log N(1/2^k)}{-\log 2^{-k}} = \lim_{k\to\infty} \frac{\log 3^k}{k \log 2} = \frac{\log 3}{\log 2}.$$