MTH6107 Chaos & Fractals

Exercises 7

Exercise 1. Let $F_k = \Phi^k([0,1]^2)$, where $\Phi(A) = \cup_{i=1}^5 \phi_i(A)$ for subsets $A \subset \mathbb{R}^2$, and where the five maps $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5$ are defined by $\phi_i(x,y) = (x_i, y_i) + (x/3, y/3)$, where $(x_1, y_1) = (0,0)$, $(x_2, y_2) = (2/3, 0)$, $(x_3, y_3) = (1/3, 1/3)$, $(x_4, y_4) = (0, 2/3)$, $(x_5, y_5) = (2/3, 2/3)$.

(a) Determine the set F_1.

(b) If F_k is expressed as a union of N_k closed squares, where the intersection of any two squares is either a singleton set or the empty set, compute the number N_k.

(c) What is the common side length of each of the N_k squares in (b) above?

(d) What is the box dimension of $F = \cap_{k=0}^\infty F_k$?

Exercise 2. Let $C_0 = [0,1]$. In the standard construction of the middle third Cantor set $C = \cap_{k=0}^\infty C_k$, describe briefly how the sets C_k are defined, and explicitly write down the sets C_1 and C_2. If C_k is expressed as a disjoint union of N_k closed intervals, compute the number N_k, and determine the common length of each of the N_k closed intervals. Use this to show that, assuming the box dimension of the middle third Cantor set C exists, then it must equal $\log 2/\log 3$

Exercise 3. If C is the middle-third Cantor set, what is the box dimension of the planar subset $C \times [0,1] = \{(x,y) : x \in C, y \in [0,1]\}$?

Exercise 4. Let ϕ_1, ϕ_2 be the iterated function system defined by the two maps $\phi_1(x) = x/10$ and $\phi_2(x) = (x+3)/10$, with $\Phi(A) := \phi_1(A) \cup \phi_2(A)$, and let C_k denote $\Phi^k([0,1])$ for $k \geq 0$. Write down the sets C_1 and C_2. If C_k is expressed as a disjoint union of N_k closed intervals, compute the number N_k, and determine the common length of each of the N_k closed intervals whose disjoint union equals C_k. Show that if the box dimension of $C = \cap_{k=0}^\infty C_k$ exists then it must equal $\log 2/\log 10$.

Exercise 5. Let ϕ_1, ϕ_2 be the iterated function system defined by the two maps $\phi_1(x) = x/10$ and $\phi_2(x) = (x+9)/10$, with $\Phi(A) := \phi_1(A) \cup \phi_2(A)$, and let C_k denote $\Phi^k([0,1])$ for $k \geq 0$. If C_k is expressed as a disjoint union of N_k closed intervals, compute the number N_k, and determine the common length of each of the N_k closed intervals whose disjoint union equals C_k. Show that if the box dimension of $C = \cap_{k=0}^\infty C_k$ exists then it must equal $\log 2/\log 10$.

Exercise 6. For C as in Exercise 5, give a description of the members of C in terms of the digits of their decimal expansion. If $f : C \to C$ is defined by $f(x) = 10x \pmod 1$ then find a point $x \in C$ which has minimal period 2 under f.

Exercise 7. Describe the construction of the *Sierpinski triangle* P, and show that if the box dimension of P exists then it must equal $\log 3/\log 2$