MTH6112 Actuarial Financial Engineering
Coursework Week 2

1. Suppose that the odds of m possible outcomes of an experiment are $o_i > 0$, where $i = 1, \ldots, m$. In other words, the return function is given by

$$r_i(j) = \begin{cases} o_i & \text{if } j = i; \\ -1 & \text{if } j \neq i. \end{cases}$$

Use the Arbitrage Theorem to show that either

$$\sum_{i=1}^{m} (1 + o_i)^{-1} = 1\,,$$

or there is an arbitrage opportunity.

2. Let $S(j)$ be a 2-period Binomial model with parameters S, u, d, r and suppose that $u > 1 + r > d$. Prove that then the risk-neutral probability is given by

$$\mathbb{P}((1,0)) = \mathbb{P}((0,1)) = pq, \quad \mathbb{P}((1,1)) = p^2, \quad \mathbb{P}((0,0)) = q^2,$$

where $p = \frac{1+r-d}{u-d}$, $q = 1 - p$.

Hints First of all, introduce the following notation: $p_{ij} = \mathbb{P}((i,j))$.

One possibility to prove the above is to consider the following two investment strategies.

Strategy 1: buy 1 share at time 0 and sell it at time 1. For each outcome (i,j), compute the return function $r_1(i,j)$. Now, use the no-arbitrage equations to prove that $p_{1,0} + p_{1,1} = p$ (and hence $p_{0,1} + p_{0,0} = q$).

Strategy 2: if $S(1) = Su$ then buy 1 share at time 1 and sell it at time 2. If $S(1) = Sd$ then don't buy anything (and hence don't earn or lose money).

Once again, for each outcome (i,j), compute the return function $r_2(i,j)$. Now, use the no-arbitrage equations and observe that they imply that $p_{1,1} = cp$ and $p_{1,0} = cq$ for some c. The rest now follows easily.

3. The price of a share follows a 3-period Binomial model $S(j)$ with parameters $S,\ u,\ d,\ r$ and suppose that $u > 1 + r > d$. A derivative on this share operates according to the following rules. If $S(2) \geq Sud$ then the owner of the derivative can buy the share at time 2 for K, where $K < Sud$. If $S(2) < Sud$ then the owner of the derivative can sell the share at time 3 for K_1, where $Su^2d \geq K_1 > Sud^2$. What is the no-arbitrage price C of this derivative?