MTH6112 Actuarial Financial Engineering
Coursework Week 11

1. An analyst is using a two-state continuous-time model to study the credit risk of zero-coupon bonds issued by different companies. The risk-neutral transition intensity function is:

   - $\widetilde{\lambda}_A(s) = 0.0148$ for Company A, and
   - $\widetilde{\lambda}_B(s) = 0.01s^2$ for Company B,

   where s measures time in years from now. The analyst observes that the credit spread on a 3 year zero-coupon bond just issued by Company B is twice that on a 3-year zero-coupon bond just issued by Company A. Given that the average recovery rate in the event of default, δ, where $0 < \delta < 1$, is the same for both companies, calculate δ. What should be the relation between the recovery rates of these two companies for there not to be arbitrage in the market?

   Remark. The credit spread on a zero-coupon bond is the difference between the yield on the bond and the yield on a similar bond issued by the government. I.e., for company i it is equal to $R_i(t,T) - r$, where r is the risk-free interest rate and

   $$R_i(t,T) = -\frac{1}{T-t}\log B(t,T).$$

   The price of a zero-coupon bond in a two-state model was derived in the Lecture and is equal to

   $$B(t,T) = e^{-r(T-t)}\left(\delta + (1-\delta)\, e^{-\int_t^T \widetilde{\lambda}(s)ds}\right).$$

2. This question is covered in the Slides of this week. Please dirty your hands and do it independently to check if you are able to calculate them.
   Let $\xi_i, i = 1, 2, \ldots, n$ be independent random variables taking the values ± 1 with probability $\mathbb{P}[\xi_1 = 1] = 1/2$.
   We denote by $\mathcal{F}_n$ the σ-algebra generated by $\xi_1, \xi_2, \ldots, \xi_n$. Further we denote

   $$S_n = \sum_{i=1}^{n} \xi_j.$$

Finally, let τ be a random variable taking values in $\mathbb{N}$, with

$$\mathbb{E}\left[\tau\right] < \infty,$$

and τ being independent of all ξ_i. Compute the following conditional expectations.

a) $\mathbb{E}[e^{\xi_1+\xi_2-\xi_3}|\xi_1, \xi_2]$

b) $\mathbb{E}[S_n|\mathcal{F}_{n-1}]$

c) $\mathbb{E}[S_n^2 - n|\mathcal{F}_{n-1}]$

d) $\mathbb{E}[e^{S_n}|\mathcal{F}_{n-1}]$

e) $\mathbb{E}[S_n^2|S_{n-1}]$

f) $\mathbb{E}[S_\tau^2|\tau]$

3. This question is also covered in the Slides of Week 11. Make sure you are able to solve it.

a) Let W_t be a standard Brownian Motion/Wiener Process and $\{\mathcal{F}_t\}_{t\geq 0}$ be a corresponding natural filtration. Let $B_t = B_0 + \mu t + \sigma W_t$ be a BM with corresponding drift and volatility. Show that W_t is a martingale.

b) Show that under the same assumption, W_t^2 is a not martingale, but $W_t^2 - t$ is.

c) Let $S_t = \mathrm{e}^{B_t}$ be a Geometric Brownian Motion starting at $S_0 = \mathrm{e}^{B_0}$ and having drift μ and volatility σ. Show that this process is not a martingale in general, but is a martingale for $\mu = -\frac{\sigma^2}{2}$.