MTH6112 Actuarial Financial Engineering
Coursework Week 9

1. Assume that the risk-free interest rate is governed by the Vasicek model. Historical data of short time risk-free interest rate is given in the table for a period January-May 2019

Date	01/01	01/02	01/03	01/04	01/05
r_t	3.56%	4.02%	3.84%	4.00%	4.18%

There are three zero-coupon bonds (see the table for their parameters) available at the market paying £1 on a corresponding maturity day.

	Issue date	Maturity date	Price on issue date
Bond 1	01/01	01/03	£0.92
Bond 2	01/02	01/04	£0.86
Bond 3	01/03	01/05	?

Find the price of Bond 3.

Solution: The formula for a zero-coupon bond price, in the context of Vasicek model, has an important property: functions A and B depend only on $\tau = T - t$ as a parameter. All three bonds given in the problem have the time period they are issued for equal to 2 months. This yields that we can write their prices as

$$B(t, T, r_t) = e^{-Ar_t + B},$$

where $A := A\left(\frac{1}{6}\right)$ and $B := B\left(\frac{1}{6}\right)$. Taking into account prices of the Bonds 1 and 2 we can set up the system of equations

$$\begin{cases} B_1 = e^{-A \cdot 0.0356 + B} & = 0.92 \\ B_2 = e^{-A \cdot 0.0402 + B} & = 0.86 \end{cases}$$

Solving the system we arrive to

$$\begin{cases} -A \cdot 0.0356 + B = & \log 0.92 \\ -A \cdot 0.0402 + B = & \log 0.86 \end{cases} \Rightarrow \begin{cases} A \cdot 0.0046 = & \log \frac{0.92}{0.86} \\ B = & \log 0.86 + A \cdot 0.0402 \end{cases}$$

$$\Rightarrow \begin{cases} A = & 14.661 \\ B = & 0.439 \end{cases}$$

By using the above we can calculate the price of Bond 3

$$B_3 = e^{-14.661 \cdot 0.0384 + 0.439} = e^{-0.124} = £0.88.$$

2. Let W_t be a standard Brownian Motion. The simplest version of the Ornstein-Uhlenbeck process X_t is defined by

$$X_t = \mathrm{e}^{-\theta t} W_{\mathrm{e}^{2\theta t}}, \quad \text{for some constant } \theta > 0.$$

a) Does this process have independent increments?

b) Is X_t a Brownian Motion?

c) What is the distribution of the increment $X_t - X_s$ for $t > s$?

d) Compute $\mu_m := \mathbb{E}[(X_t)^m]$ for all integer $m > 0$.

e) Compute $\mathrm{Cov}\{X_t, X_s\}$.

Solution

a) Increments $X_{t_{i+1}} - X_{t_i}$ can be expressed in terms of the Brownian motion as follows:

$$\begin{aligned} X_{t_{i+1}} - X_{t_i} &= \mathrm{e}^{-\theta t_{i+1}} W_{\mathrm{e}^{2\theta t_{i+1}}} - \mathrm{e}^{-\theta t_i} W_{\mathrm{e}^{2\theta t_i}} \\ &= \mathrm{e}^{-\theta t_{i+1}} \left(W_{\mathrm{e}^{2\theta t_{i+1}}} - W_{\mathrm{e}^{2\theta t_i}} \right) + \left(\mathrm{e}^{-\theta t_{i+1}} - \mathrm{e}^{-\theta t_i} \right) W_{\mathrm{e}^{2\theta t_i}}. \end{aligned}$$

It is clear that the first term in this expression is independent from all previous history of the Brownian motion (see properties of a Brownian motion). However, the second one is not. To formally prove that the increments are not independent, let us take three different times $t < s < r$ and calculate the covariance $\mathrm{Cov}[X_r - X_s, X_s - X_t]$.

$$\begin{aligned} \mathrm{Cov}[X_r - X_s, X_s - X_t] &= \mathrm{Cov}[X_r, X_s] + \mathrm{Cov}[X_s, X_t] \\ &\quad - \mathrm{Cov}[X_r, X_t] - \mathrm{Cov}[X_s, X_s] \\ &= \mathrm{e}^{-\theta r - \theta s} \mathrm{Cov}[W_{\mathrm{e}^{2\theta r}}, W_{\mathrm{e}^{2\theta s}}] + \mathrm{e}^{-\theta t - \theta s} \mathrm{Cov}[W_{\mathrm{e}^{2\theta t}}, W_{\mathrm{e}^{2\theta s}}] \\ &\quad - \mathrm{e}^{-\theta t - \theta r} \mathrm{Cov}[W_{\mathrm{e}^{2\theta t}}, W_{\mathrm{e}^{2\theta r}}] - \mathrm{e}^{-2\theta s} \mathrm{Var}[W_{\mathrm{e}^{2\theta s}}] \\ &= \mathrm{e}^{\theta s - \theta r} + \mathrm{e}^{\theta t - \theta s} - \mathrm{e}^{\theta t - \theta r} - 1 \\ &= \left(\mathrm{e}^{-\theta r} - \mathrm{e}^{-\theta s} \right) \left(\mathrm{e}^{\theta s} - \mathrm{e}^{\theta t} \right) > 0. \end{aligned}$$

b) It follows from the above that X_t does not have independent increments, so it is not a Brownian Motion.

c) An increment $X_t - X_s$ can be written as a sum of two independent Gaussian random variables. Indeed

$$X_t - X_s = \mathrm{e}^{-\theta t}\left[W_{\mathrm{e}^{2\theta t}} - W_{\mathrm{e}^{2\theta s}}\right] + \left(\mathrm{e}^{-\theta t} - \mathrm{e}^{-\theta s}\right)\left[W_{\mathrm{e}^{2\theta s}} - W_0\right].$$

Thus, $X_t - X_s$ is a Gaussian random variable with mean zero and variance

$$\mathrm{Var}\left[X_t - X_s\right] = \mathrm{e}^{-2\theta t}\left(\mathrm{e}^{2\theta t} - \mathrm{e}^{2\theta s}\right) + \left(\mathrm{e}^{-\theta t} - \mathrm{e}^{-\theta s}\right)^2 \mathrm{e}^{2\theta s} = 2\left(1 - \mathrm{e}^{-\theta(t-s)}\right).$$

d) For the m-th moment we use the formula derived in lectures:

$$\mu_m = \mathbb{E}\left[\mathrm{e}^{-m\theta t} W^m_{\mathrm{e}^{2\theta t}}\right] = \begin{cases} 0, & \text{if } m \text{ is odd,} \\ \mathrm{e}^{-2p\theta t}\frac{(2p)!}{2^p p!}\mathrm{e}^{2\theta t p} = \frac{(2p)!}{2^p p!}, & \text{if } m = 2p \text{ (is even).} \end{cases}$$

Interestingly, these moments do not depend on t, the time of the process.

e) Finally, for the covariance (which in fact has been computed above) one has:

$$\mathrm{Cov}\left[X_t, X_s\right] = \mathrm{e}^{-\theta t - \theta s}\mathrm{Cov}\left[W_{\mathrm{e}^{2\theta t}}, W_{\mathrm{e}^{2\theta s}}\right] = \mathrm{e}^{-\theta t - \theta s}\mathrm{e}^{2\theta \min(t,s)} = \mathrm{e}^{-\theta|t-s|}.$$

The covariance vanishes on as $|t - s| \to \infty$ and thus the values Y_t and Y_s of the OU process are almost independent when the time distance between them is large.

Remark. One can extract from the above an interesting property of the Ornstein-Uhlenbeck (OU) process.

Note that $\mathbb{E}(X_t) = 0$ and $\mathrm{Var}(X_t) = 1$ (derive this fact from the above results!). Also, its increments are Gaussian random variables whose variance is bounded from above. This means that typically, even at a large time scale, the OU process would stay at a bounded distance from its average. In contrast, the BM often deviates from its mean value (that is, from 0) by a distance "proportional" to $\sqrt{t}$.