Actuarial Financial Engineering

Week 9

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February 1, 2023

Overview of this week

14. Interest Rates Term Structure

14. Interest Rates Term Structure

- Interest rate modelling is the **most important** topic in derivative pricing.
- Interest rate derivatives account for around **80%** of the value of derivative contracts outstanding, mainly **swaps** and **credit derivatives** used to support the **securitisation** of debt portfolios.

Question: Why is modelling interest rates more complicated than modelling share prices?

14. Interest Rates Term Structure

Question: Why is modelling interest rates more complicated than modelling share prices?

Answer: Because interest rates depend not only on the **current time**, but also on the **term of the investment**.
For example, an investor with a 10-year bond will normally earn a different rate of interest than an investor with a 5-year bond.

14. Interest Rates Term Structure

In this section we will look at models for the **term structure of interest rates**.

Different from Actuarial Mathematics I or Financial Mathematics I, in particular, we will focus on models that are

- stochastic,
- framed in continuous time,
- and arbitrage-free.

14. Interest Rates Term Structure

There are **two main types** of models used to describe interest rates mathematically:

- The Heath-Jarrow-Morton approach uses an Ito process to model the **forward rate** for an investment with a fixed maturity. We will not consider this approach in this module.
- Short-rate models use an Ito process to model the **short rate**. We will look at three specific models of this type:
  - the Vasicek model,
  - the Cox-Ingersoll-Ross model,
  - and the Hull-White model.

Ito processes are a key feature of these models, so it might be helpful to review the topics of Brownian motion, Ito's Lemma and stochastic differential equations.

14. Interest Rates Term Structure

Fixing $t = 0$ and plotting yield, $R(0, T)$ or $r(0, T)$, against maturity T, gives the **yield curve** which gives information on the **term structure**.

Definition 14.1 (Interest Rates Term Structure)

$\sim$ shows how **interest rates** for different **maturities** are related.
$\sim$ is a function of interest rates on maturities, e.g. $R(0, T)$ or $r(0, T)$

Question: Typically, the yield curve increases with maturity. Why?

14. Interest Rates Term Structure

Question: Typically, the yield curve increases with maturity. Why?

Reflecting uncertainty about far-future rates. However, if current rates are unusually high, the yield curve can be downward sloping, and is inverted.

Question: If $R(0, T)$ is independent of T, what is the shape of the term structure?

14. Interest Rates Term Structure

Various theories explaining the shape of the yield curve:

- The expectations theory: the **long-term rate** is determined purely by current and future expected **short-term rates**, so that the expected final value of investing in **a sequence of short-term bonds** equals the final value of wealth from investing in **long-term bonds**.
- The market segmentation theory: different agents in the market have different **objectives**: pension funds determine longer-term rates, market makers determine short-term rates, and businesses determine medium-term rates, which are all determined by the **supply and demand of debt** for these different market segments.
- The liquidity preference theory: **lenders** want to lend **short term** while borrowers wish to borrow long term, and so forward rates are higher than expected future zero rates (and yield curves are upward sloping).

14. Interest Rates Term Structure

Theorem 14.1

If the term structure is deterministic, then the No-Arbitrage Principle implies that

$$P(0, T) = P(0, t)P(t, T). \tag{1}$$

Proof: Hint: what happens if $P(0, T) < P(0, t)P(t, T)$, or $P(0, T) > P(0, t)P(t, T)$? What strategy can you adopt to gain profit without taking any risk? Please refer to the hand-written notes during the lecture.

14. Interest Rates Term Structure

Proof:

If $P(0,T) < P(0,t)P(t,T)$,

(a) Buy a bond maturing at time T,

(b) and write a fraction $P(t,T)$ of a bond maturing at t. This gives $PV = P(0,t)P(t,T) - P(0,T) > 0$ at time 0.

(c) At time t settle the written bonds, raising the required sum of P(t,T) by issuing a single unit bond maturing at T. At time T close the position, retaining the initial profit.

If $P(0,T) > P(0,t)P(t,T)$, then adopt the opposite strategy. □

	0	t	T
(a)	$-P(0,T)$		1
(b)	$P(0,t) \cdot P(t,T)$	$-P(t,T)$	
(c)		$P(t,T)$	-1

14.1. Desirable characteristics of a term structure model

Question: What are term structure models used for?

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Answer:
The main uses of term structure (interest rate) models are:

- by **bond traders** looking to identify and exploit **price inconsistencies**
- for calculating the **price of interest rate derivatives**
- by **investors** with a portfolio involving bonds or loans who want to set up a **hedged position**
- for **asset-liability modelling**.

14.1. Desirable characteristics of a term structure model

There are two types of term structure models:

Equilibrium models

- start with a theory about the economy,
- such that interest rates revert to some long-run average,
- are positive or their volatility is constant or geometric,
- are based on the model for (typically) the short rate.
- e.g. Rendleman and Bartter, Vasicek and Cox-Ingersoll-Ross.
- Based on 'economic fundamentals', equilibrium models rarely reproduce observed term structures, which is unsatisfactory.

No-arbitrage models

14.1. Desirable characteristics of a term structure model

There are two types of term structure models:
Equilibrium models
No-arbitrage models

- use the term structure as an input
- and are formulated to adhere to the no-arbitrage principle.
- An example of a no-arbitrage model is the Hull-White (one-and two-factor).

14.1. Desirable characteristics of a term structure model

Question: What characteristics of a term structure model are regarded as desirable features?

14.1. Desirable characteristics of a term structure model

Characteristics of a term structure model that are regarded as desirable features:

- The model should be **arbitrage free**.
- Interest rates should ideally be **positive**. E.g. Vasicek model allow interest rates to go negative.
- $r(t)$ and other interest rates should exhibit some form of **mean-reverting** behaviour.
- Computationally **easy to calculate** the prices of bonds and certain derivative contracts.
- Produce **realistic dynamics**. E.g. Does it give rise to a full range of plausible yield curves, i.e. upward-sloping, downward-sloping and humped?
- With appropriate parameter estimates, fit **historical** interest rate data adequately.
- Can be calibrated easily to **current market data**
- **Flexible** enough to cope properly with a range of derivative contracts.

14.2. The Vasicek Model (1977)

In the real world markets, interest rates behave, at a local scale, in a way which resembles some kind of a Brownian motion.
In fact, this random process is a **function of a Brownian motion** and has interesting and important properties which are very different from those of the Brownian motion.

14.2. The Vasicek Model (1977)

The simplest model describing this behaviour is the so called Vasicek Model(1977).

Definition 14.2

Mathematically speaking, the Vasicek Model is the Ornstein-Uhlenbeck process:

$$dr = -a(r - \mu)dt + \sigma dW_t \tag{2}$$

where $a > 0, \mu > 0$ and we usually think of σ as a positive number.

Remark:

It is not important whether σ is positive or negative.
The reason for that is the if $dW_t = W_{t+dt} - W_t$, then $\sigma dW_t \sim \mathcal{N}(0, \sigma^2 dt)$ and $-\sigma dW_t \sim \mathcal{N}(0, \sigma^2 dt)$
– they have the same distribution.

14.2. The Vasicek Model (1977)

Properties of the Vasicek Model

We need the following two theorems to prove the properties of Vasicek Model.

Theorem 13.4

Suppose that $r(t)$ is a random process which satisfies the equation

$$dr = -a(r-\mu)dt + \sigma dW_t.$$

Then

$$r(t) = b + (r(0)-\mu)e^{-at} + \sigma e^{-at}\int_0^t e^{as}dW_s.$$

Theorem 12.3

$$\int_0^t f(s)dW_s \sim \mathcal{N}\Big(0, \int_0^t (f(s))^2 ds\Big). \tag{3}$$

14.2. The Vasicek Model (1977)

Property 1: explicit solution

First of all, we already know the explicit solution according to Theorem 13.4:

$$r(t) = (r_0 - \mu)e^{-at} + \mu + \sigma e^{-at} \int_0^t e^{as} dW_s \,. \tag{4}$$

14.2. The Vasicek Model (1977)

Property 2: what happens if $t \to 0$?

Next, according to the Theorem 12.3, $\int_0^t e^{as} dW_s \sim \mathcal{N}(0, \int_0^t e^{2as} ds)$ and so we can compute $\mathbb{E}(r(t))$ and $\mathrm{Var}(r(t))$. Namely,

$$\mathbb{E}(r(t)) = (r_0 - \mu)e^{-at} + \mu. \tag{5}$$

Since $e^{-at} \to 0$ as $t \to \infty$, we see that also

$$\mathbb{E}(r(t)) \to \mu \quad \text{as } t \to \infty. \tag{6}$$

14.2. The Vasicek Model (1977)

Property 2 (cont.):

Next

$$\begin{aligned}\mathrm{Var}(r(t)) &= \mathrm{Var}\Big(\sigma e^{-at}\int_0^t e^{as}dW_s\Big) = \sigma^2 e^{-2at}\mathrm{Var}\Big(\int_0^t e^{as}dW_s\Big)\\ &= \sigma^2 e^{-2at}\int_0^t e^{2as}ds\\ &= \frac{\sigma^2}{2a}e^{-2at}(e^{2at}-1)\end{aligned}$$

Thus

$$\mathrm{Var}(r(t)) = \frac{\sigma^2}{2a}(1-e^{-2at}) \tag{7}$$

and

$$\mathrm{Var}(r(t)) \to \frac{\sigma^2}{2a} \quad \text{as } t\to\infty \tag{8}$$

Remark: (6) and (8) are due to $a > 0$.

14.2. The Vasicek Model (1977)

Property 3: distribution of $r(t)$

Hence

$$r(t) \sim \mathcal{N}\left((r_0 - \mu)e^{-at} + \mu, \frac{\sigma^2}{2a}(1 - e^{-2at})\right)$$

and for large values of t, $r(t) \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{2a}\right)$ (with good precision).

This means that for large values of t, the distribution of $r(t)$ **does not depend on** t.

14.2. The Vasicek Model (1977)

Property 4: $r(t)$ can be negative

The *unfortunate* property of this model is that $r(t)$ can be negative.
However, the probability of such an event is small when σ is small.

Question: Assuming Property 3 of Vasicek model in the lecture slides, what is the probability that $r(t) < 0$ for large values of t? More precisely, compute $\lim_{t\to\infty} \mathbb{P}(r(t) < 0)$.

14.2. The Vasicek Model (1977)

Property 4 (cont.)

Solution:

$$\lim_{t\to\infty} \mathbb{P}(r(t) < 0) = \Phi\left(\frac{-\mu\sqrt{2a}}{|\sigma|}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{-\mu\sqrt{2a}}{|\sigma|}} e^{-\frac{x^2}{2}} dx.$$

Can you now see that this probability decreases to 0 as $|\sigma| \to 0$?

14.2. The Vasicek Model (1977)

Property 5: mean reversion

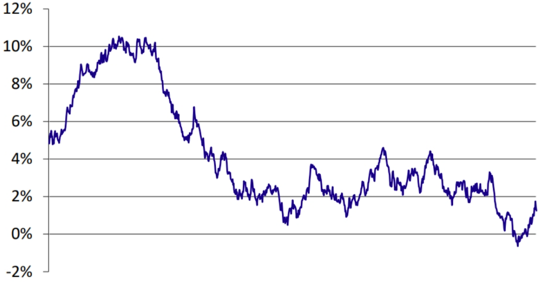
The most important good feature of this model is the "mean reversion" property of $r(t)$: the value $r(t)$ will eventually return to its long-term mean μ.

$$dr(t) = -a(r(t) - \mu)dt + \sigma dW_t.$$

If $r(t) - \mu > 0$, then the larger the deviation of $r(t)$ from μ, the stronger is the "drive" $-a(r(t) - \mu)dt$ which pushes $r(t)$ back to μ.
If $r(t) - \mu < 0$, then $-a(r(t) - \mu)dt > 0$ and again the interest rate r is pushed back to μ.

14.2. The Vasicek Model (1977)

The graph below show a simulation of this process based on the parameter values
$\alpha = 0.1$, $\mu = 0.06$ and $\sigma = 0.02$.
Please check the five properties one by one.



Source: the Actuarial Education Company.

14.3. The Cox-Ingersoll-Ross Model (1985)

The following two models describing the behaviour of interest rates are in many ways more advanced than the Vasicek model.

Definition 14.3

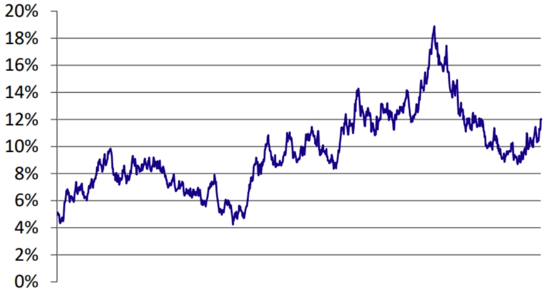
The Cox-Ingersoll-Ross model (CIR) with parameters $\alpha > 0,\ \mu > 0,\ \sigma > 0$ is the one according to which the interest rate is governed by the following equation:

$$dr(t) = -\alpha(r(t) - \mu)dt + \sigma\sqrt{r(t)}dW_t.$$

The following important property of this model can be proved: if $\sigma^2 < 2\alpha\mu$ then $r(t) > 0$, i.e. $r(t)$ can be strictly **positive** if σ^2 is small enough.

14.3. The Cox-Ingersoll-Ross Model (1985)

The graph below shows a simulation of this process based on the parameter values $\alpha = 0.1$, $\mu = 0.06$ and $\sigma = 0.1$.



Source: the Actuarial Education Company.

14.4. The Hull-White Model (1990)

Definition 14.4

The Hull-White model is the one which assumes that the interest rate is governed by the following equation:

$$dr(t) = -\alpha(r(t) - \mu(t))dt + \sigma dW_t, \tag{9}$$

where $\mu(t) > 0$ is a given deterministic function of t, $\alpha > 0$, $\sigma > 0$.

The difference between this model and the Vasicek model is that here $\mu(t)$ depends on t (whereas in the Vasicek model μ is a constant, $\mu(t) = \mu$.)

14.5. Summary of short-rate modelling

Model	Dynamics	$r_t > 0$ for all t	Distribution of r_t
Vasicek	$dr_t = \alpha(\mu - r_t)dt + \sigma d\tilde{W}_t$	No	Normal
CIR	$dr_t = \alpha(\mu - r_t)dt + \sigma\sqrt{r_t}\, d\tilde{W}_t$	Yes	Non-central chi-squared
Hull-White	$dr_t = \alpha(\mu(t) - r_t)dt + \sigma d\tilde{W}_t$	No	Normal

14.5. Summary of short-rate modelling

The following table summarises the characteristics of the Vasicek, Cox-Ingersoll-Ross (CIR) and Hull-White models.

	Vasicek	CIR	Hull-White
Arbitrage-free	Yes	Yes	Yes
Positive interest rates	No	Yes	No
Mean-reverting interest rates	Yes	Yes	Yes
Easy to price bonds and derivatives	Yes	Yes(1)	Yes
Realistic dynamics	No(2)	No(2)	No(2)
Adequate fit to historical data	No	No	Yes
Easy to calibrate to current data	No	No	Yes(3)
Can price a wide range of derivatives	No(4)	No(4)	No(4)

Notes:

(1) Although the CIR model is harder to use than the other two models, it is more tractable than models with two or more factors.

(2) All three models produce perfectly correlated changes in bond prices, which is inconsistent with the empirical evidence, and fail to model periods of high and low interest rates and high and low volatility.

(3) Whilst one-factor models are generally difficult to calibrate, the Hull-White model is easier than the other two because its time-varying mean-reversion function aids fitting.

(4) All three models can be used to price short-term, straightforward derivatives, but not complex derivatives.

14.5. Summary of short-rate modelling

One-factor models (a model in which interest rates are assumed to be influenced by a single source of randomness), such as Vasicek and CIR, have certain limitations.

Bearing in mind that the purpose of interest rate models is to price interest rate derivatives, there are some **short-comings** of short-rate models:

- Single factor short-rate models mean that all maturities behave in the same way - there is no independence.
- There is little consistency in valuation between the models.
- They are difficult to calibrate.