MTH6112 Actuarial Financial Engineering
Coursework Week 5

You may need the following theorems to solve the questions.

Theorem 1 *Suppose that conditions (i), (ii), (iii) of Theorem 1 in Coursework Week 4 are satisfied and in addition dividend is paid continuously at rate q and is reinvested in the underlying asset.*

Then the price $C_q(S,t)$ of the derivative with payoff (exercise) time t is given by

$$C_q(S,t) = e^{-rt}\mathbb{E}\left(R(\tilde{S}(t))\right), \quad \text{where} \ \ \tilde{S}(t) = Se^{\tilde{\mu}t+\sigma W(t)} \ \ \text{and } \tilde{\mu} = r - q - \frac{\sigma^2}{2}.$$

Corollary We could equivalently say that $C_q(S,t) = C(e^{-tq}S,t)$.

Theorem 2 *Suppose that conditions (i), (ii), (iii) of Theorem 1 in Coursework Week 4 are satisfied and in addition the proportionate dividend $D = dS(t_0)$ is payed at time t_0.*

Then the price $C_2(S,t,d)$ of the derivative exercised at time t is computed as follows:

(a)If $t \le t_0$ then

$$C_2(S,t,d) = e^{-rt}\mathbb{E}\left(R(\tilde{S}(t))\right), \quad \text{where} \ \ \tilde{S}(t) = Se^{\tilde{\mu}t+\sigma W(t)} \ \ \text{and } \tilde{\mu} = r - \frac{\sigma^2}{2}.$$

(b)If $t > t_0$ then

$$C_2(S,t,d) = e^{-rt}\mathbb{E}\left(R(\tilde{S}(t))\right), \quad \text{where} \ \ \tilde{S}(t) = (1-d)Se^{\tilde{\mu}t+\sigma W(t)} \ \ \text{and } \tilde{\mu} = r - \frac{\sigma^2}{2}.$$

Remark Equivalently, $C_2(S,t,d) = C(S,t)$ if $t \le t_0$ and $C_2(S,t,d) = C((1-d)S,t)$ if $t > t_0$.

1. Recall the Question 2 of Coursework Week 4. Suppose that the price $S(t)$ of a share is described by the GBM with parameters $S,\ \mu,\ \sigma,\ r$.

   Suppose now that the above share provides a dividend yield of rate q which is paid continuously and is reinvested in the share. What is the price C of the derivative with the same payoff function?

2. Recall the Question 3 of Coursework Week 4. Suppose again that the price $S(t)$ of a share is described by the GBM with parameters $S,\ \mu,\ \sigma,\ r$.

   Consider an option with expiration time T and payoff function given by

$$R(S(T)) = \begin{cases} K & \text{if } S(T) < K, \\ 0 & \text{if } S(T) \geq K. \end{cases}$$

   (Note that if a portfolio consists of 1 share and 1 such option then the payoff of at least £K is guaranteed.)

   (a) Suppose now that the above share provides a dividend yield of rate q which is paid continuously and is reinvested in the share. What is the price C_q of the derivative with the same payoff function?

   (b) Suppose that a discrete proportionate dividend of rate d is paid at time $T/2$ and is immediately reinvested in the share. The expiration time of the option is t, $0 < t \leq T$. Write down the formulae for the price of this option in the following 2 cases: $t \leq T/2$ and $T/2 < t \leq T$.

3. Suppose that the price of a share is $S(t)$, $0 \leq t \leq T$. Suppose also that a discrete proportional dividend is paid at time t_0 at rate d. Prove that if $S(t) > (1-d)S(t_0)$ for all $t \in (t_0, t_0 + \epsilon)$, where $\epsilon > 0$, then there is an arbitrage opportunity.

4. Suppose that the price $S(t)$ of the share is driven by a geometric Brownian motion with parameters $S,\ \mu,\ \sigma$, that is $S(t) = Se^{\mu t + \sigma W(t)}$. Suppose also that a proportional dividend on this share is paid continuously at rate $q > 0$ and is reinvested in the share. The continuously compounded interest rate is r. Compute the no-arbitrage price of a derivative with the payoff function $R(T) = \frac{1}{T}\int_0^T S(t)dt$.

5. A company's share is currently traded at the price of £18.49. A dividend on this share is paid continuously at rate q and is reinvested in the share. Two options are available on the market with the same strike price of £18 and the same maturity time of 6 months. European Call option is worth of £1.72 and a European Put option is priced at £1.52. Assuming the continuously compounded interest rate is 16%, find the dividend rate.
   Hint: The Call-Put parity for a dividend paying share is

$$C - P = Se^{-qT} - Ke^{-rT}.$$

Please compare it with the Call-Put parity you learnt in Financial Mathematics 1. We will further discuss it in Week 6.