Actuarial Financial Engineering

Week 5

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Overview of this week

7. Hedging within the framework of the MPBM and the GBM model

We have considered hedging in the context of the **One-Period Binomial Model**.
We shall see that the formulae derived there will be instrumental for the analysis of **Multi-Period Binomial Model (MPBM)**.

7.1. Hedging within the framework of the MPBM

We start with the **statement of the problem**.

Suppose that the price of a share $S(j),\ 0 \le j \le n$, follows a MPBM with parameters s, u, d, r.
As usual, the no-arbitrage condition $d < 1 + r < u$ is supposed to be satisfied.

Recall that there are $n+1$ possible values of $S(n)$: sd^n, $sud^{n-1}, \ldots, su^i d^{n-i}, \ldots, su^n$.

Consider a derivative on this share with the payoff time n and the payoff function V. We shall consider payoff functions which depend only on $S(n)$ - the price of the share at time n, that is

$$V(su^i d^{n-i}) = V_{n,i},\ 0 \le i \le n.$$

In words: if at time n the price of the share is $S(n) = su^i d^{n-i}$ then the owner of the derivative is paid $V_{n,i}$. (The price goes up i times)

Denote by C the no arbitrage price of this derivative.

7.1. Hedging within the framework of the MPBM

Question 1. What is the price C of this derivative?

Question 2. How should the trader who **sells** this derivative invest the C so that at time n, s/he is able to meet the obligation to payoff $V_{n,i}$ whatever the i, $0 \leq i \leq n$, turns out to be?

7.1. Hedging within the framework of the MPBM

Answer to Question 1.
The price C of such a derivative is given by

$$C = (1+r)^{-n} \sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i} V_{n,i}, \text{ where } p = \frac{1+r-d}{u-d}, \ q = \frac{u-1-r}{u-d}.$$

7.1. Hedging within the framework of the MPBM

Proof:

A sequence of prices $S(1), ..., S(n)$ with $S(n) = su^i d^{n-i}$ has the risk-neutral probability (r-np for short) $p^i q^{n-i}$ (the price goes up i times and it goes down $n - i$ times).
Each such sequence is uniquely defined by i time moments $j_1, ..., j_i$ such that $S(j_k) = S(j_{k-1})u$.
The total number of these sequence is equal to $\binom{n}{i}$ which is the number of possibilities to choose i time-moment out of n.
Therefore, the r-np $\tilde{\mathbb{P}}(S(n) = su^i d^{n-i}) = \binom{n}{i} p^i q^{n-i}$.
Hence also $\tilde{\mathbb{P}}(V = V_{n,i}) = \binom{n}{i} p^i q^{n-i}$ and $\tilde{\mathbb{E}}(V) = \sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i} V_{n,i}$.
By Theorem 5.2.

$$C = (1+r)^{-n}\tilde{\mathbb{E}}(V) = (1+r)^{-n} \sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i} V_{n,i}$$

$\square$

7.1. Hedging within the framework of the MPBM

We shall now **answer Question 2** which is the main goal of this section.

At each time moment $j \geq 0$, the trader is allowed to buy shares and to deposit money into the bank.
It is natural to expect that the number of shares and the total capital the trader must have at time j depends on $S(j) = su^i d^{j-i}$ - the value of the price of the underlying share.

We thus introduce the following notation: given that $S(j) = su^i d^{j-i}$,
$X_{j,i}$ denotes the number of shares in the portfolio at time j,
$V_{j,i}$ the total value of the trader's portfolio at time j.
Then $V_{j,i} - X_{j,i} su^i d^{j-i}$ is the amount of money that should be deposited in the bank at time j.

Exercise: Draw the timeline.

7.1. Hedging within the framework of the MPBM

Lemma 7.1

For each j, $0 \le j \le n-1$, the values $(X_{j,i},\ V_{j,i})$, $0 \le i \le j$, are computed as follows.

$$V_{j,i} = (1+r)^{-1}\left(pV_{j+1,i+1} + qV_{j+1,i}\right), \quad X_{j,i} = \frac{V_{j+1,i+1} - V_{j+1,i}}{S(j)(u-d)} \tag{1}$$

7.1. Hedging within the framework of the MPBM

Proof:

Recall that $V_{j+1,i+1}$ and $V_{j+1,i}$ is the capital the seller of the derivative must have at time $j+1$
in order to be able to meet the payoff obligations at time n
given that $S(j+1) = su^{i+1}d^{j-i}$ and $S(j+1) = su^i d^j$ respectively.

If $S(j) = su^i d^{j-i}$ then either $S(j+1) = S(j)u = su^{i+1}d^{j-i}$ or $S(j+1) = S(j)d = su^i d^j$ and hence $V_{j,i}$ must be such that the capital at time $j+1$ becomes $V_{j+1,i+1}$ and $V_{j+1,i}$ respectively.

Hence we are in the setting of the one-period binomial model (OPBM) and Equation (1) are the same as the corresponding formulae for OPBM (we Equation (3) and (5)in the slides of Week 2): we replace $V_1,\ V_2,\ P$ in these formulae by $V_{j+1,i+1},\ V_{j+1,i},\ V_{j,i}$ respectively). $\square$

7.1. Hedging within the framework of the MPBM

$X_{0,0}, \ V_{0,0}$:

We thus can do the following: Step 1. Use (1) to compute $X_{n-1,i}$ and $V_{n-1,i}$, $0 \leq i \leq n-1$, from known values $V_{n,i}$, $0 \leq i \leq n$.

Step 2. Similarly, compute $X_{n-2,i}$ and $V_{n-2,i}$, $0 \leq i \leq n-2$, from $V_{n-1,i}$, $0 \leq i \leq n-1$

Continue these calculations until, moving from the values found for time $j+1$ to values at time j. After n steps, we shall compute $X_{0,0}, \ V_{0,0}$.

7.1. Hedging within the framework of the MPBM

The hedging strategy now works as follows.

1. At time $t = 0$, buy $X_{0,0}$ shares and deposit $V_{0,0} - X_{0,0}s$ (the rest of the initial capital) into the bank.

2. At time $t = 1$, your total capital is $V_{0,0}S(1) + (V_{0,0} - X_{0,0}s)(1 + r)$ and is either $V_{1,1}$ or $V_{1,0}$ (depending on whether $S(1) = su$ or $S(1) = sd$.)
Use it to buy the corresponding number of shares (which will be either $X_{1,1}$ or $X_{1,0}$ respectively) and deposit the rest into the bank.

3. At each next time moment, say j you act as follows.
If the price $S(j) = su^i d^{j-i}$ then the total value of your portfolio is $V_{j,i}$.
Use it to buy $X_{j,i}$ shares and deposit $V_{j,i} - X_{j,i}S(j)$ into the bank.

7.1. Hedging within the framework of the MPBM

Question.
What is the relationship between C and $V_{0,0}$?

7.1. Hedging within the framework of the MPBM

Remark.
Note that $C = V_{0,0}$. Indeed, we have two investments.

Investment 1: buy the derivative for C.
Investment 2: Invest $V_{0,0}$ in the way prescribed by the hedging strategy.

Both investments produce the same result at time n. Hence, by the Law of One Price, their cost should be the same.

7.2. Hedging within the framework of the GBM

Throughout this section we suppose that
the **price of a share** follows the Geometric Brownian Motion (GBM)
$S(t) = S\mathrm{e}^{\mu t+\sigma W(t)},\ t \geq 0,$
where $S,\ \mu,\ \sigma$ are the parameters of the GBM,
$W(t)$ is the standard Brownian motion.
The interest rate **compounded continuously** is r.

7.2. Hedging within the framework of the GBM

We consider a derivative on this asset with a payoff function $G(S(T))$ and the payoff time T.

We know (Theorem 5.3) that the price C of such a derivative is given by

$$C = \mathrm{e}^{-rT}\mathbb{E}[G(\tilde{S}(T))] = \mathrm{e}^{-rT} \int_{-\infty}^{\infty} G\left(S(0)\mathrm{e}^{\tilde{\mu}T+\sigma\sqrt{T}x}\right) f(x)\mathrm{d}x, \tag{2}$$

where $f(x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}$ is the p.d.f. of the standard normal random variable.

7.2. Hedging within the framework of the GBM

Question. Can the seller of the derivative invest C so that to be able to meet the payoff obligation at time T?
The seller is allowed to buy the underlying shares and to deposit the money into the bank.

The answer is: Yes, by continually reinvesting the portfolio.

7.2. Hedging within the framework of the GBM

To make this answer precise, we have to answer two more questions.
Question (a). What should be the **total value of the portfolio** at time t, $0 \leq t \leq T$?
Question (b). How many **shares** should be in the portfolio at time t, $0 \leq t \leq T$?

7.2. Hedging within the framework of the GBM

Answer to Question (a). The total value of the portfolio should be equal to the price $C(S(t), t)$ of the derivative at time t, given that the price of the share is $S(t)$.

We thus have introduced a new notation, $C(S(t), t)$ – the price of the derivative which is a function of two variables, $S(t)$ and t. The important fact is that we can compute this price.

Namely, since the conditions of the market allow the investor to buy the derivative at any time t, $0 \leq t \leq T$, $C(S(t), t)$ can be computed in the same way as C in (2). The only difference is that the starting price of the share now is $S(t)$ and the payoff takes place in $T - t$ units of time. Thus

$$C(S(t), t) = \mathrm{e}^{-r(T-t)} \int_{-\infty}^{\infty} G\left(S(t)\mathrm{e}^{\mu(T-t)+\sigma\sqrt{T-t}x}\right) f(x)\mathrm{d}x. \tag{3}$$

7.2. Hedging within the framework of the GBM

Answer to Question (b). The number of shares in the portfolio at time t should be

$$\Delta(t) = \frac{\partial C(S,t)}{\partial S}, \quad \text{where} \quad S = S(t). \tag{4}$$

Remark. (4) explains why the term Delta-hedging is used in finance.

7.2. Hedging within the framework of the GBM

The hedging strategy is:

- At each time t, have $\Delta(t)$ shares and keep $[C(S(t), t) - \Delta(t)S(t)]$ in the bank.
- This can be achieved by re-investing the total capital in our portfolio at times $0,\ h,\ 2h, \dots,\ T$, where h as a short period of time as possible.
- At time 0, we own $C(S(0))$ and use them to buy $\Delta(0)$ shares and deposit the rest of the capital in the bank.
- At time h, we change the number of shares in the portfolio to $\Delta(h)$ and deposit the rest of the capital in the bank. We can do that because we know the price $S(h)$ (from our observation of the development in the market of shares) and we can use $S(h)$ to carry out the necessary calculations.
- The same operation is then repeated ta times $2h$, $3h$, ..., T.
- If h is very small then this process looks like a continual reinvestment of the portfolio.
- This strategy is designed so that to make sure that at time T the value of our portfolio is $R(S(T))$ which means we can meet our financial obligations.

7.2. Hedging within the framework of the GBM

Example. If, in the above formulae, $G(x) = (x - K)^+$ then this is case of the call option $\mathrm{Call}(K, T)$. In this case we know the explicit expressions for $C(S, t)$ and $\Delta(t)$. Namely, using Black-Scholes formula, we obtain that the value of the portfolio at time t should be

$$C(S(t), t) = S(t)\Phi(\omega(t)) - K\mathrm{e}^{-r(T-t)}\Phi(\omega(t) - \sigma\sqrt{T-t}) \,, \tag{5}$$

where

$$\omega(t) = \frac{\ln \frac{S(t)}{K} + r(T-t)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}$$

and $\Phi(x)$ is the cumulative distribution function of a standard Normal random variable, that is

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-u^2/2}\, du \,.$$

7.2. Hedging within the framework of the GBM

Example (cont.). Thus, in this example the number of shares $\Delta(t)$ which should be in the portfolio at time t is

$$\Delta(t) = \frac{\partial C(S,t)}{\partial S}\Big|_{S=S(t)} = \Phi(\omega(t)). \tag{6}$$

The last formula is due to the property we mentioned last week in Section 6 The Greeks, $\frac{\partial C}{\partial S} = \Phi(\omega)$.

Finally, we have a simple expression for the amount Y(t) of money that should be deposited in the bank:

$$Y(t) = C(S(t),t) - S(t)\Delta(t) = -K\mathrm{e}^{-r(T-t)}\Phi(\omega(t) - \sigma\sqrt{T-t}).$$

7.2. Hedging within the framework of the GBM

Remark.
In this example, $\Delta(t)$ is always strictly positive, while the cash $Y(t)$ is strictly negative. This means that at each moment t the amount $-Y(t)$ is borrowed from the bank and invested in shares (together with $C(S(t), t)$).

7.2. Hedging within the framework of the GBM

Explanation of the reason for (4) (not examinable):
To simplify the calculations and the formulae, suppose that $\mu = 0$, that is $S(t) = \mathrm{e}_{\sigma W(t)}$.
If at time t the price of the share is $S(t)$ then at time $t+h$ the price will be, approximately, either $S(t)\mathrm{e}^{\sigma\sqrt{h}}$ or $S(t)\mathrm{e}^{-\sigma\sqrt{h}}$. So, the price of the derivative at time $t+h$ will be either $C(S(t)\mathrm{e}^{\sigma\sqrt{h}}, t+h)$ or $C(S(t)\mathrm{e}^{-\sigma\sqrt{h}}, t+h)$. We know from the discussion of hedging for the One-Period Binomial Model that the number of shares that should be in the portfolio at time t is given by

$$\frac{C(S(t)\mathrm{e}^{\sigma\sqrt{h}}, t+h) - C(S(t)\mathrm{e}^{-\sigma\sqrt{h}}, t+h)}{S(t)(\mathrm{e}^{\sigma\sqrt{h}} - \mathrm{e}^{-\sigma\sqrt{h}})}.$$

As $h \to 0$, we obtain

$$\Delta(t) = \lim_{h\to 0}\left[\frac{C(S(t)\mathrm{e}^{\sigma\sqrt{h}}, t+h) - C(S(t)\mathrm{e}^{-\sigma\sqrt{h}}, t+h)}{S(t)(\mathrm{e}^{\sigma\sqrt{h}} - \mathrm{e}^{-\sigma\sqrt{h}})}\right] = \frac{\partial C(S,t)}{\partial S}, \quad \text{where } S = S(t).$$

7.2. Hedging within the framework of the GBM

Remark.
The proof of the last relation easily follows from the fact that $F(b) - F(a) = F'(\xi)(b-a)$, where $\xi \in (a, b)$. You should know this formula from the Calculus I course. Applying it to the difference in the numerator of the fraction we obtain

$$C(S\mathrm{e}^{\sigma\sqrt{h}}, t+h) - C(S\mathrm{e}^{-\sigma\sqrt{h}}, t+h) = \left.\frac{\partial C(S, t+h)}{\partial S}\right|_{S=\xi} (S\mathrm{e}^{\sigma\sqrt{h}} - S\mathrm{e}^{-\sigma\sqrt{h}}),$$

where $\xi \in (S\mathrm{e}^{\sigma\sqrt{h}}, S\mathrm{e}^{-\sigma\sqrt{h}})$. It is now obvious that

$$\Delta(t) = \lim_{h\to 0}\left[\left.\frac{\partial C(S, t+h)}{\partial S}\right|_{S=\xi}\right] = \frac{\partial C(S, t)}{\partial S}, \quad \text{where} \quad S = S(t).$$

8. Dividends

So far, we have not taken into account a very important aspect of the behaviour of financial assets - the existence of dividends.

What is a dividend?

Definition 8.1

Dividend is a sum of money
paid regularly by a company to its shareholders
out of its profits (or reserves).

We want to answer the following two questions concerned with assets paying dividends.
Question 1. What are the different types of dividends and what are the ways in which the dividends can be modelled mathematically?
Question 2: What is the price of an option on an asset paying dividends?

8.1. Examples of models for dividend payments

The following examples answer Question 1 in a way which is sufficient for the purposes of this module.

1. **Discrete absolute dividends**: at times $t_1 < \cdots < t_n$ a certain known amount $D_1, \ldots, D_n$ is paid.
2. **Discrete proportional dividends**: at times $t_1 < \cdots < t_n$ the amount $d_1 S(t_1), \ldots, d_n S(t_n)$ is paid, where $S(t)$ is the price of the underlying share at time t.
3. **Continuous proportional dividends**: If $\Delta t > 0$ is a small time interval, then from time t to $t + \Delta t$ the amount paid is $qS(t + \Delta t)\Delta t$ per each share, where $q > 0$. **Exercise.** Usually q is between 0.02 - 0.05. Explain why q can't be lager than 1.

The dividends are usually paid either in cash or in shares.
We shall always suppose that a dividend is re-invested in the underlying share (which is just another way of saying that it is paid in shares).

8.2. Continuous dividend rates: how many shares do we own?

Suppose that dividend is paid continuously as described in Example 3 and is re-invested in the underlying share:
if between times t and $t + \Delta t$ the dividend is $qS(t + \Delta t)\Delta t$ per share,
then this money is used to buy shares for the price of $S(t + \Delta t)$ per share (which is simply its market price at time $t + \Delta t$).
Our model is now completely defined and the following lemma tells us how many shares we shall own at time t if at time 0 we had 1 share.

Lemma 8.1

Suppose that the dividend is paid continuously and is reinvested in the underlying share as described above.
Denote by $N(t)$ the number of shares in the portfolio at time t. Then $N(t) = \mathrm{e}^{qt}$.

8.2. Continuous dividend rates: how many shares do we own?

Proof of the lemma is not examinable.

Proof:

If $N(t)$ is the number of shares we have at time t, then at time $t + \Delta t$, where $\Delta t > 0$ is a very small positive number, they cost $N(t)S(t + \Delta t)$. We thus are paid $qN(t)S(t + \Delta t)\Delta t$ and hence the number of shares we buy is

$$\frac{qN(t)S(t + \Delta t)\Delta t}{S(t + \Delta t)} = qN(t)\Delta t.$$

But then the number of shares we have at time $t + \Delta t$ is

$$N(t + \Delta t) = N(t) + N(t)q\Delta t.$$

8.2. Continuous dividend rates: how many shares do we own?

Proof (cont.):
Hence

$$\frac{N(t+\Delta t) - N(t)}{\Delta t} = qN(t) \tag{7}$$

and as $\Delta t \to 0$, we obtain

$$N'(t) = qN(t) \ .$$

We know that this differential equation has the following general solution:

$$N(t) = A\mathrm{e}^{qt}, \quad \text{where } A \ \text{ is any constant.}$$

We also know that $N(0) = 1$ and hence $1 = A$. We thus see that $N(t) = \mathrm{e}^{qt}$.$\square$

8.2. Continuous dividend rates: how many shares do we own?

Remarks.

- The above proof is not rigorous because, strictly speaking, the dividend paid (per share) for the period from t to $t + \Delta t$ is

$$q \cdot S(t + \Delta t)\Delta t + o(\Delta t),$$

where $o(\Delta t)$ (pronounced "o small of delta t") is much smaller than Δt in the following sense:

$$\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0.$$

Taking this into account, one can make the above proof completely rigorous. However, you are not required to do that in this module.

8.2. Continuous dividend rates: how many shares do we own?

Remarks.

- Note that we don't use any specific properties of $S(t)$ such as it being a geometric Brownian motion. So this lemma is an example of a statement which is *model-independent*.
- *Exercise*. Explain the following statement: if at tome $t = 0$ the number of shares in the portfolio is k then the number of shares at time $t > 0$ is $k\mathrm{e}^{qt}$ and the total cost of these shares is $k\mathrm{e}^{qt}S(t)$.

8.3. Risk-neutral process for an asset with proportional continuously paid dividend

The goal of this section is to answer Question 2.

Let us state this question more precisely:
given that dividends are paid continuously at rate q
and that the continuously compounded interest rate is r,
what is, at time $t = 0$, the price C of a derivative with the payoff function $R(S(t))$?

8.3. Risk-neutral process for an asset with proportional continuously paid dividend

We know (see Theorem 5.2 from Slides Week 3-4) that

$$C = \mathrm{e}^{-rt}\tilde{\mathbb{E}}(R(S(t))), \tag{8}$$

where $\tilde{\mathbb{E}}$ denotes the expectation over the risk-neutral probability.
To be able to use (8), we have to know this risk-neutral probability.
This probability depends on the choice of a concrete model for $S(t)$.

8.3. Risk-neutral process for an asset with proportional continuously paid dividend

However, we start with a fact which will be used later and which does not depend on the choice of the model.

Lemma 8.2

Consider a share with continuously paid dividend of rate q which is reinvested into the share.

Let $S(0)$ be the price of 1 share at time $t = 0$ and let $M(t)$ be the cost at time $t > 0$ of the portfolio which at time $t = 0$ consists of 1 share.
Suppose that the continuously compounded interest rate is r. Then

$$\tilde{\mathbb{E}}(M(t)) = S(0)\mathrm{e}^{rt}, \tag{9}$$

where the expectation $\tilde{\mathbb{E}}$ is taken over the risk-neutral probability.

8.3. Risk-neutral process for an asset with proportional continuously paid dividend

Proof.

Obviously, $M(t) = N(t)S(t) = \mathrm{e}^{qt}S(t)$.

If we decide to sell our shares at time t then $M(t)$ is the payoff we get.

Our return at time t is $M(t) - S(0)\mathrm{e}^{rt}$ and hence, by the Arbitrage Theorem the following relations hold:

$$\tilde{\mathbb{E}}(M(t) - S(0)\mathrm{e}^{rt}) = 0 \;\text{ or, equivalently, }\; \tilde{\mathbb{E}}(M(t)) = S(0)\mathrm{e}^{rt}.$$

$\square$

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Price: $S(0)$ $\qquad$ $S(t) = S(0)\mathrm{e}^{rt}$

Cost/Payoff: $M(0)$ $\qquad$ $M(t) = \mathrm{e}^{qt}S(t)$

8.3. Risk-neutral process for an asset with proportional continuously paid dividend

Corollary 8.1

Suppose that the conditions of Lemma 8.2 are satisfied. Then

$$\tilde{\mathbb{E}}(S(t)) = S(0)\mathrm{e}^{(r-q)t}. \tag{10}$$

Proof.

Equation (9) can be rewritten as

$$\tilde{\mathbb{E}}(\mathrm{e}^{qt}S(t)) = S(0)\mathrm{e}^{rt} \tag{11}$$

which implies $\mathrm{e}^{qt}\tilde{\mathbb{E}}(S(t)) = S(0)\mathrm{e}^{rt}$. Dividing both sides of the last equality by e^{qt}, we obtain (10).□

8.3. Risk-neutral process for an asset with proportional continuously paid dividend

Let us return to the question about the risk-neutral probability.
The answer depends on the choice of the model and we assume $S(t)$ follow the geometric Brownian motion.

Theorem 8.1

Suppose that
(a) The price of a share follows the geometric Brownian motion with parameters $S,\ \mu,\ \sigma$, that is $S(t) = S\mathrm{e}^{\mu t+\sigma W(t)}$. The continuously compounded interest rate is r.
(b) The dividend is paid continuously at rate q and is reinvested in the share;
Then the risk-neutral probability is the one corresponding to the GBM given by

$$\tilde{S}(t) = S\mathrm{e}^{\tilde{\mu}t+\sigma W(t)}, \quad \textit{where} \ \ \tilde{\mu} = r - q - \frac{\sigma^2}{2}.$$

8.3. Risk-neutral process for an asset with proportional continuously paid dividend

Before discussing the proof of this theorem, we state the following corollary.

Corollary 8.2

Suppose that that the conditions of Theorem 8.1 are satisfied. Consider a derivative with the payoff function $R(S(t))$ and the payoff time is t. Then

$$\tilde{\mathbb{E}}(R(S(t))) = \mathbb{E}(R(\tilde{S}(t)))$$

and the price of this derivative is given by

$$C = \mathrm{e}^{-rt}\mathbb{E}(R(\tilde{S}(t))) = \mathrm{e}^{-rt}\int_{-\infty}^{\infty} R(S\mathrm{e}^{\tilde{\mu}t+\sigma\sqrt{t}x})f(x)dx,$$

where $f(x)$ is the standard normal density: $f(x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}$.

8.3. Risk-neutral process for an asset with proportional continuously paid dividend

Corollary 8.2 (cont.):

In particular, if $R(\tilde{S}(T)) = (S(T) - K)^+$ the we obtain the price of a $\mathrm{Call}(K, T)$:

$$C = \mathrm{e}^{-rT}\mathbb{E}(S(T) - K)^+ = \mathrm{e}^{-rT}\int_{-\infty}^{\infty}(S\mathrm{e}^{\tilde{\mu}T + \sigma\sqrt{T}x} - K)^+ f(x)dx.$$

8.3. Risk-neutral process for an asset with proportional continuously paid dividend

Proof of Theorem 8.1.

The statement of the theorem can be divided into two parts. Namely,
Statement 1: The risk-neutral process has the form $\tilde{S}(t) = S\mathrm{e}^{\tilde{\mu}t+\sigma W(t)}$.
Statement 2: $\tilde{\mu}$ is as stated in the theorem: $\tilde{\mu} = r - q - \frac{\sigma^2}{2}$.
Statement 1 requires a relatively difficult proof and we shall simply believe it. However, we shall use it in order to prove the second statement.
The idea is to compute $\tilde{\mathbb{E}}(S(t))$ in the way suggested by Statement 1 and to compare the result with the one in (10).
Namely, the first statement of the theorem means that in order to compute $\tilde{\mathbb{E}}(S(t))$ we have to replace $S(t)$ by $\tilde{S}(t)$ and then carry out the computation of the usual expectation $\mathbb{E}(\tilde{S}(t))$. So

$$\tilde{\mathbb{E}}(S(t)) = \mathbb{E}(\tilde{S}(t)) = \mathbb{E}(S\mathrm{e}^{\tilde{\mu}t+\sigma W(t)}) = S\mathrm{e}^{\tilde{\mu}t}\mathbb{E}(\mathrm{e}^{\sigma W(t)})$$

8.3. Risk-neutral process for an asset with proportional continuously paid dividend

Proof of Theorem (cont.).

We know (see Slides Week 1) that

$$\mathbb{E}(\mathrm{e}^{\sigma W(t)}) = \mathrm{e}^{\frac{\sigma^2}{2}t} \text{ and hence } \tilde{\mathbb{E}}(S(t)) = S\mathrm{e}^{\tilde{\mu}t}\mathrm{e}^{\frac{\sigma^2}{2}t} = S\mathrm{e}^{(\tilde{\mu}+\frac{\sigma^2}{2})t}.$$

Comparing this with (10), we obtain

$$S\mathrm{e}^{(\tilde{\mu}+\frac{\sigma^2}{2})t} = S\mathrm{e}^{(r-q)t}$$

and therefore $\tilde{\mu} + \frac{\sigma^2}{2} = r - q$ and finally

$$\tilde{\mu} = r - q - \frac{\sigma^2}{2}.$$

□

8.4. One application of Theorem 8.1: computing the price of the call option

Suppose that the price of a share follows the GBM, $S(t) = S\mathrm{e}^{\mu t+\sigma W_t}$.
Suppose also that the dividend is paid continuously at rate q.
What is the price of a European $\mathrm{Call}(K, t)$?

8.4. One application of Theorem 8.1: computing the price of the call option

Denote by $C_q(S, t, K, \sigma, r)$ the price of the European call option on a share paying dividend at rate q as above
and by $C(S, t, K, \sigma, r)$ the price of the standard European call option (no dividend is paid).

Theorem 8.2

Suppose that
1. *The price of a share is* $S(t) = S\mathrm{e}^{\mu t + \sigma W_t}$.
2. *The interest rate compounded continuously is* r.
3. *The dividend is paid continuously at rate* q. *Then*

$$C_q(S, t, K, \sigma, r) = C(\mathrm{e}^{-qt} S, t, K, \sigma, r).$$

8.4. One application of Theorem 8.1: computing the price of the call option

Proof.

In both cases, the payoff function is $R((S(t)) = (S(t) - K)^+$ (the standard payoff for the European call option).

Hence, by the general formula (see Theorem 5.2 in Slides Week 3-4)

$$C_q(S, t, K, \sigma, r) = \mathrm{e}^{-rt}\tilde{\mathbb{E}}(S(t) - K)^+.$$

By Theorem 8.1, $\tilde{\mathbb{E}}(S(t) - K)^+ = \mathbb{E}(\tilde{S}(t) - K)^+$ and we obtain

$$\begin{aligned}
C_q(S, t, K, \sigma, r) &= \mathrm{e}^{-rt}\mathbb{E}(\tilde{S}(t) - K)^+ \\
&= \mathrm{e}^{-rt}\mathbb{E}\left[\left(S\mathrm{e}^{(r-q-\frac{\sigma^2}{2})t+\sigma W_t} - K\right)^+\right] \\
&= \mathrm{e}^{-rt}\mathbb{E}\left[\left((S\mathrm{e}^{-qt})\mathrm{e}^{(r-\frac{\sigma^2}{2})t+\sigma W_t} - K\right)^+\right]
\end{aligned}$$

8.4. One application of Theorem 8.1: computing the price of the call option

Proof. (cont.)
We also know that if $S(t) = \bar{S}\mathrm{e}^{\mu t + \sigma W_t}$ and no dividend is paid then

$$C \equiv C(\bar{S}, t, K, \sigma, r) = \mathrm{e}^{-rt}\mathbb{E}\left(\bar{S}\mathrm{e}^{(r-\frac{\sigma^2}{2})t+\sigma W_t} - K\right)^+ .$$

Comparing the formulae for C_q and C we see that if $\bar{S} = \mathrm{e}^{-qT}S_0$ then $C_q(S, t, K, \sigma, r) = C(\mathrm{e}^{-qt}S, t, K, \sigma, r)$. $\square$

8.5. Discrete proportional dividends

We shall now turn to a particular case of Example 2 defined in section 8.1.

Recall that, according to this model, at times $0 \leq t_1 < t_2 < ... < t_n$ the dividends $d_1 S(t_1)$, $d_2 S(t_2)$, ..., $d_n S(t_n)$ are paid.
Here $S(t)$ is the price of the underlying share (or,more generally, asset) and $0 < d_j < 1$ for all j, $1 \leq j \leq n$.

In the following model we will discuss $n = 1$. So:
1. At time t_0 the amount $dS(t_0)$ is paid.
2. The dividend $dS(t_0)$ is reinvested in the underlying share ('immediately' after time t_0), where $0 < d < 1$.

8.5. Discrete proportional dividends

Question: What happens to the price of the share shortly after t_0?

It is natural to suggest that, for very small $\epsilon > 0$, during the period of time between (t_0 and $t_0 + \epsilon$), the price of the share will drop to $S(t_0) - dS(t_0) = (1-d)S(t_0)$.
One explanation of this statement is that if the share "gives away" a part of its cost, its price drops by this amount.
One can state this more precisely: we suggest that

$$\lim_{\epsilon > 0,\, \epsilon \to 0} S(t_0 + \epsilon) = S(t_0) - dS(t_0) = (1-d)S(t_0).$$

8.5. Discrete proportional dividends

The following exercise provides a more fundamental reason for this suggestion.

Exercise.

1. Prove that if $S(t) > (1-d)S(t_0)$ for all $t \in [t_0, t_0+\epsilon)$, where $\epsilon > 0$, then there is an arbitrage opportunity.
2. Prove that if $S(t) < (1-d)S(t_0)$ for all $t \in [t_0, t_0+\epsilon)$, where $\epsilon > 0$, then there is an arbitrage opportunity.

8.5. Discrete proportional dividends

Solution to part 2 of the Exercise.
First, recall that to *short-sell* a share means:
(a) Borrow a share from, say, a bank, and sell it (you can now use the money you get).
(b) At the time determined by the contract, return *the share (not its cost!) together with the dividend* the share would have paid to its owner.

Now, the arbitrage is achieved as follows.
1. Short-sell the share at time $t = t_0$ for $S(t_0)$.
2. At time $\bar{t}$, $t_0 < \bar{t} < t_0 + \epsilon$ buy the share for $S(\bar{t})$ and return it to the owner of the share together with the dividend $d\,S(t_0)$ the owner would have been paid.
Your return now is

$$S(t_0) - S(\bar{t}) - d\,S(t_0) = (1-d)S(t_0) - S(\bar{t}) > (1-d)S(t_0) - (1-d)S(t_0) = 0.$$

You thus have not invested any of your money but got a positive return (and that is the arbitrage). $\square$

8.5. Discrete proportional dividends

(Ignore interest rate because ϵ is very small.)

	0	t_0	$\bar{t}$	$t_0 + \epsilon$
Cash flow:		$S(t_0)$	$-S(\bar{t}) - d\,S(t_0)$	

8.5. Discrete proportional dividends

The following lemma summarizes several simple properties of the price of a share paying discrete proportional dividend.

Lemma 8.3

1. *Shortly after the dividend has been paid, the share price is* $(1-d)S(t_0)$. *More precisely,*
$$\lim_{t \geq t_0,\, t \to t_0} S(t) = (1-d)S(t_0).$$
2. *When we reinvest the dividend in shares, we buy* $\frac{d}{1-d}$ *additional shares (the reinvestment takes place straight after* t_0*!)*
3. *The value of the portfolio (consisting of 1 share) is* $S(t)$ *at time* $t \leq t_0$ *and it is* $\frac{1}{1-d}S(t)$ *at time* $t > t_0$.

8.5. Discrete proportional dividends

Proof.

1. We have explained statement 1 above.
2. For a very small positive ϵ the price of the share at time $t_0 + \epsilon$ is $S(t_0 + \epsilon) = (1 - d)S(t_0)$ and $d\,S(t_0)$ is the amount we reinvest. Hence the number of shares we buy is
$$\frac{d\,S(t_0)}{S(t_0 + \epsilon)} = \frac{d\,S(t_0)}{(1 - d)S(t_0)} = \frac{d}{1 - d}.$$
3. When $t \leq t_0$, we have 1 share and its cost is $S(t)$ (which is the value of the portfolio).
When $t > t_0$, we have $1 + \frac{d}{1-d} = \frac{1}{1-d}$ shares and so the value of the portfolio is $\frac{1}{1-d}S(t)$.

8.6. Risk-neutral process for an asset paying discrete proportional dividend

Note that all previous statements of concerning the model with a discrete proportional dividend were **model-independent**.
The risk-neutral process cannot be model-independent.
The following theorem describes this process for the case when the price of an asset follows a GBM.

8.6. Risk-neutral process for an asset paying discrete proportional dividend

Theorem 8.3

Suppose that

1. *The price of an asset follows the geometric Brownian motion,* $S(t) = S\mathrm{e}^{\mu t + \sigma W(t)}$.
2. *Discrete proportional dividend is paid at time* t_0 *at rate* d, $0 < d < 1$.
3. *The continuously compounded interest rate is* r.

Then the risk-neutral process is

$$\tilde{S}_1(t) = \begin{cases} \tilde{S}(t), & \text{if } t \le t_0 \\ (1-d)\tilde{S}(t), & \text{if } t > t_0. \end{cases}$$

Where $\tilde{S}(t) = S\mathrm{e}^{\tilde{\mu} t + \sigma W(t)}$ *and* $\tilde{\mu} = r - \frac{\sigma^2}{2}$.

8.6. Risk-neutral process for an asset paying discrete proportional dividend

This theorem allows one to compute the expectations with respect to the risk-neutral probability in the usual way, e.g $\tilde{\mathbb{E}}(R(S(t))) = \mathbb{E}(R(\tilde{S}_1(t)))$.

Exercise.
Write down, in terms of the integral the formula for the price of a European $\mathrm{Call}(K, t)$ for $t > t_0$ and for $t \leq t_0$.