MTH6112 Actuarial Financial Engineering
Coursework Week 4

You may need the following theorem to solve the questions.

Theorem 1 *Suppose that:*

(i) The price of an asset is driven by a GBM with parameters $S,\ \mu,\ \sigma$, *that is* $S(t) = Se^{\mu t+\sigma W(t)}$.

(ii) The interest rate compounded continuously is r.

(iii) A derivative (on the asset) has a payoff function $R(S(t))$ *with payoff time* t.

Then the no-arbitrage price of this derivative is given by

$$C(S,t) = e^{-rt}\mathbb{E}\left(R(\tilde{S}(t))\right), \quad \textit{where} \ \ \tilde{S}(t) = Se^{\tilde{\mu}t+\sigma W(t)} \ \ \textit{and} \ \tilde{\mu} = r - \frac{\sigma^2}{2}.$$

Remark The price of the derivative depends on all parameters. However, we deliberately emphasize the dependence on two parameters, the initial price S of the asset and the time t at which the option expires / is exercised.

1. The prices of the stock of F. Bancroft & Sons follow a geometric Brownian motion with parameters $\mu = 0.15$ and $\sigma = 0.21$. Presently, the stock's price is 38 pounds. Consider a call option having three months until its expiration time and having a strike price of 41 pounds.

   (a) What is the probability that the call option will be exercised?

   (b) If the interest rate is 5%, what is the Black-Scholes price of the call?

2. Suppose that the price $S(t)$ of a share is described by the GBM with parameters $S,\ \mu,\ \sigma,\ r$.

   Consider a derivative on this share which provides at time T a payoff $R(S(T)) = S(T)^2 + 1$. Compute the no-arbitrage price C of this derivative.

   Remark. This is an example of an artificial 'option'. The advantage of this example is that the answer can be expressed explicitly in terms of the parameters of the model.

3. Suppose again that the price $S(t)$ of a share is described by the GBM with parameters $S,\ \mu,\ \sigma,\ r$.

   Consider now an option with expiration time T and payoff function given by

$$R(S(T)) = \begin{cases} K & \text{if } S(T) < K, \\ 0 & \text{if } S(T) \geq K. \end{cases}$$

   (Note that if a portfolio consists of 1 share and 1 such option then the payoff of at least £K is guaranteed.)

   (a) Consider the case when no dividend is paid. Compute the no-arbitrage price of this option.

   (b) Consider again the case when no dividend is paid and the expiration time is T.

   If you are the seller of this option, what should be your hedging strategy? Namely, how many shares must be in your portfolio and how much money should be deposited in the bank at time t, $0 \leq t \leq T$, in order for you to be able to meet your obligation at time T?

4. Compute $\mathbb{E}\left(e^{aW(t)+bW(t+s)}\right)$, where $t > 0$ and $s > 0$.

5. The price of a share $S(t)$ evolves according to a Geometric Brownian Motion with parameters $S,\ \mu,\ \sigma$. The continuously compounded interest rate is r. A derivative on this option has the payoff function

$$R(T) = \frac{1}{T}\int_0^T S(t)S(T)dt.$$

   The payoff time is T. What is the no-arbitrage price of this derivative?

   Hint Use the result obtained in the former question.