Actuarial Financial Engineering

Week 3-4

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Overview of this week

4. CLT and GBM

The Binomial model can be viewed as an **approximation** of a continuous model according to which the price of an asset (say share) is governed by the Geometric Brownian Motion (GBM).

In this section, we first state an important version of the Central Limit Theorem (CLT) and then use it to explain how the GBM arises as a natural choice of a model describing the behaviour of the price of an asset.

4. CLT and GBM

Recall that the GBM is a model according to which the price of an asset (say, a share) is given by $S(t) = S(0)e^{\mu t + \sigma W_t}$,
where W_t is the standard Wiener process and μ, σ are the drift and the volatility parameters of the Brownian motion $Y(t) = \mu t + \sigma W_t$.

Our aim is to show that the random process $S(t)$ can be obtained as a limit of a sequence of discrete processes describing the behaviour of the prices.
More precisely, we shall construct a sequence of binomial models $S_n(t)$ which converge to $S(t)$ as $n \to \infty$.

4.1 Central Limit Theorem

Theorem 4.1 (Central Limit Theorem)

Let $Y_1, Y_2, \cdots, Y_n$ *be a sequence of i.i.d. random variables,*
$\mathbb{E}(Y_n) = a$, $\mathrm{Var}(Y_n) = \sigma^2$, $|Y_n| < C$.
Then the sequence

$$Z_n = \frac{\sum_{j=1}^{n} Y_j - na}{\sqrt{n}\sigma}$$

converges in distribution, as $n \to \infty$*, to the standard normal random variable.*

4.1 Central Limit Theorem

Central Limit Theorem (cont.):

Namely

1. For all x $\lim_{n\to\infty} \mathbb{P}(Z_n < x) = \Phi(x)$, where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-\frac{y^2}{2}}\, dy$.
2. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying, for some constants $c_1 > 0,\ c_2 > 0$, the estimate $|g(x)| < c_1 \mathrm{e}^{c_2|x|}$. Then

$$\lim_{n\to\infty} \mathbb{E}(g(Z_n)) = \mathbb{E}(g(Z))$$

where $Z \sim \mathcal{N}(0,1)$ (has the standard normal distribution). In other words, we have

$$\lim_{n\to\infty} \mathbb{E}(g(Z_n)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \mathrm{e}^{-\frac{x^2}{2}}\, dx$$

4.1 Central Limit Theorem

- The two statements in this theorem are equivalent (each of them implies the other one).
- The proof of the above theorem as well as of the equivalence of its two statements is beyond the scope of this course. However, we are going to use this theorem and you are required to know its statement and to be able to apply it as it is done in the proofs explained below.

4.2 Why does the GBM describe the behaviour of the prices?

Recall that the GBM is a model according to which the price of an asset (say, a share) is given by $S(t) = S(0)e^{\mu t + \sigma W_t}$,
where W_t is the standard Wiener process and μ, σ are the drift and the volatility parameters of the Brownian motion $Y(t) = \mu t + \sigma W_t$.

Our aim is to show that the random process $S(t)$ can be obtained as a limit of a sequence of discrete processes describing the behaviour of the prices.
More precisely, we shall construct a sequence of binomial models $S_n(t)$ which converge to $S(t)$ as $n \to \infty$.

4.2.1. Construction of the approximating binomial models

The construction of the process approximating our GBM is carried out in two steps.

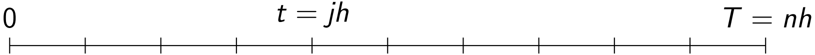
Step 1. Divide $[0, T]$ into n equal intervals of length $h = \frac{T}{n}$ and define for each time of the form jh the value $S_n(jh)$ as follows:

1. At time $t = 0$, $S_n(0) = S(0)$, where $S(0)$ is the same as in the GBM.
2. At time $t = jh$, $1 \leq j \leq n$, set

$$S_n(jh) = S(0)\mathrm{e}^{\mu hj+\sigma\sqrt{h}(Y_1+Y_2+\dots Y_j)} = S(0)\mathrm{e}^{\mu t+\sigma\sqrt{h}\sum_{k=1}^{j} Y_k}, \tag{1}$$

where $Y_1,\ Y_2, \dots, Y_n, \dots$ is a sequence of independent identically distributed random variables each taking either value 1 or -1 with probability $\frac{1}{2}$:

$$\mathbb{P}(Y_j = 1) = \frac{1}{2}, \quad \mathbb{P}(Y_j = -1) = \frac{1}{2}.$$



4.2.1. Construction of the approximating binomial models

Step 2. For $t \in [0, T]$ set

$$S_n(t) = S_n(jh) \text{ where } j \text{ is such that } jh \leq t < (j+1)h \ . \tag{2}$$

Remarks.

1. For every fixed n, the sequence of prices $S_n(jh)$, $j = 0,\, 1, 2,\, ...,\, n$, evolves according to a binomial model.
Indeed, $S_n(jh) = S_n((j-1)h)\mathrm{e}^{\mu h + \sigma \sqrt{h} Y_j}$.
We can therefore say that

$$\text{either} \quad S_n(jh) = S_n((j-1)h)u \quad \text{or} \quad S_n(jh) = S_n((j-1)h)d, \tag{3}$$

where $u = \mathrm{e}^{\mu h + \sigma \sqrt{h}}$ (which corresponds to $Y_j = 1$) and $d = \mathrm{e}^{\mu h - \sigma \sqrt{h}}$ (which corresponds to $Y_j = -1$).
As we know, relations (3) is the one defining the binomial model.

4.2.1. Construction of the approximating binomial models

Remarks. (cont.)
2. The additional feature of this binomial model:
each sequence of prices $S_n(jh)$, $j = 0,\ 1, 2,\ \dots,\ n$ occurs with a certain real life probability, namely with probability 2^{-n} (prove this statement!).
This fact plays a very important role for the theorems we discuss in this section.

4.2.1. Construction of the approximating binomial models

Remarks. (cont.)
3. We thus suppose that at each time the change of the price is driven by the product of two factors: one is $\mathrm{e}^{\mu h}$ and $\mathrm{e}^{\pm\sigma\sqrt{h}}$.

- $\mathrm{e}^{\mu h}$: is not random and would usually be pushing the price up: it is natural to expect that $\mu > 0$ (can you explain why?).
- $\mathrm{e}^{\pm\sigma\sqrt{h}}$: is random and can push the price both up and down with the same probability.

Note that when h is small, $\sqrt{h}$ is much larger than h (in the sense that $\frac{\sqrt{h}}{h} \to \infty$ as $h \to 0$).
So, **at each particular step** the influence of the second factor is much stronger than that of the first one. However, **in a long run**, with probability which is close to one, the first factor accumulates a much stronger influence on the price because of the cancelations in the second sum.

4.2.1. Construction of the approximating binomial models

Figure: Source: LCV Advisors

4.2.2. Convergence of Binomial approximations to the GBM

Now we show the convergence of Binomial approximations to the GBM.

Theorem 4.2 ($S_n(t) \to S(t)$)

When $n \to \infty$, the process $S_n(t) \to S(t)$ in the following sense:
if $g(z_1, \cdots, z_k)$ is a "good enough" function of k variables (say, continuous, and growing no faster than exponentially in each variable)
and
$0 < t_1 < t_2 \cdots < t_k \leq T$ are time moments,
then

$$\lim_{n\to\infty} \mathbb{E}[g(S_n(t_1), S_n(t_2), \cdots S_n(t_k))] = \mathbb{E}[g(S(t_1), S(t_2), \cdots S(t_k))] \tag{4}$$

4.2.2. Convergence of Binomial approximations to the GBM

Proof.

We prove this theorem only for the case $k = 1$, and so we can simplify our notation and write t in place of t_1. We thus have to show that if $0 < t \leq T$ then, as $n \to \infty$,

$$\mathbb{E}(g(S_n(t))) \to \mathbb{E}(g(S(t))) \overset{(CLT\ 2.)}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(S(0)\mathrm{e}^{\mu t + \sigma\sqrt{t}x})\mathrm{e}^{-\frac{x^2}{2}} dx.$$

Note that according to (2), for every t there is a j such that $jh \leq t < (j+1)h$. To simplify (slightly) the further steps of the proof, suppose that $t = jh$. Then $h = \frac{t}{j}$ and

$$S_n(t) \overset{Step2, Eq(2)}{=} S_n(jh) \overset{Eq(1)}{=} S(0)\mathrm{e}^{\mu jh + \sigma\sqrt{h}(Y_1 + Y_2 + \ldots Y_j)} \overset{(*)}{=} S(0)\mathrm{e}^{\mu t + \sigma\sqrt{\frac{t}{j}}\sum_{k=1}^{j} Y_k},$$

where $(*)$ is due to the fact that we can replace jh by t and $\sqrt{h}$ by $\sqrt{\frac{t}{j}}$. Now, $h \to 0$ when $n \to \infty$ and therefore also $j \to \infty$.

4.2.2. Convergence of Binomial approximations to the GBM

Proof (cont.).
Since Y_i are independent with $\mathbb{E}(Y_j) = 1 \times \frac{1}{2} - 1 \times \frac{1}{2} = 0$ and $\mathrm{Var}(Y_j) = \mathbb{E}(Y_j^2) - (\mathbb{E}Y_j)^2 = 1$, we can now make use of CLT in order to be able to control the behaviour of $\sqrt{\frac{t}{j}} \sum_{k=1}^{j} Y_k$.
Namely, according to CLT

$$\sqrt{\frac{t}{j}} \sum_{k=1}^{j} Y_k = \sqrt{t} \times \frac{1}{\sqrt{j}} \sum_{k=1}^{j} Y_k \to \sqrt{t} Z \text{ as } \ j \to \infty.$$

where Z is the standard normal random variable, $Z \sim \mathcal{N}(0,1)$ and the convergence is understood as explained by CLT ($n = j$ in Theorem 4.1).

4.2.2. Convergence of Binomial approximations to the GBM

Proof (cont.).
For a “good” function g we have

$$\mathbb{E}\left(g\left(\frac{1}{\sqrt{j}}\sum_{k=1}^{j} Y_k\right)\right) \to \mathbb{E}(g(Z)), \quad \text{where} \quad Z \sim N(0,1).$$

In particular, as $n \to \infty$,

$$\mathbb{E}(g(S_n(t))) \overset{Eq(5)}{=} \mathbb{E}\left(g(S(0)\mathrm{e}^{\mu t+\sigma\sqrt{t}\frac{1}{\sqrt{j}}\sum_{k=1}^{j} Y_k})\right) \to \mathbb{E}\left(g(S(0)\mathrm{e}^{\mu t+\sigma\sqrt{t}Z})\right)$$

$$\overset{CLT\ 2.}{=} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(S(0)\mathrm{e}^{\mu t+\sigma\sqrt{t}x})\mathrm{e}^{-\frac{x^2}{2}}\,dx. \square$$

5.1.1. The assumptions underlying the B-S model

1. The price of the underlying share follows a Geometric Brownian Motion (GMB), i.e. $dS(t) = S(t)(\mu dt + \sigma dW_t)$
2. There are no risk-free arbitrage opportunities.
3. The risk-free rate of interest is constant, the same for all maturities and the same for borrowing or lending.(Not critical and can be relaxed.)
4. Unlimited short selling (that is, negative holdings) is allowed.
   So, we are allowed to sell unlimited amounts of securities that we do not own.
5. There are no taxes or transaction costs.
   (This is important since we will need to continuously rebalance some risk-free portfolios.)
6. The underlying asset can be traded continuously and in infinitesimally small numbers of units.
   (Infinite divisibility of securities is necessary to ensure that perfect hedges can be achieved.)

5.1.1. The assumptions underlying the B-S model

The key general implication of the underlying assumptions is that the **market** in the underlying share is **complete**:
all derivative securities have payoffs which can be **replicated**.

This consequence is at odds with the real world and implies problems with the underlying assumptions.

Exercise: List the main defects of the assumptions of the Black-Scholes model.

5.1.1. The assumptions underlying the B-S model

Despite all of the potential flaws in the model assumptions, analyses of market derivative prices indicate that the Black-Scholes model does give **a very good approximation** to the market.

It is worth stressing here that all models are only approximations to reality.
It is always possible to take a model and show that its underlying assumptions do not hold in practice.

This does not mean that a model has no use.
A model is useful if, for a specified problem, it provides answers which are a good approximation to reality or if it provides insight into underlying processes.

5.1.2. Generalization of the B-S model

Recall:

$$S_n(jh) = S(0)\mathrm{e}^{\mu hj + \sigma\sqrt{h}(Y_1 + Y_2 + \dots Y_j)} = S(0)\mathrm{e}^{\mu t + \sigma\sqrt{h}\sum_{k=1}^{j} Y_j}, \tag{5}$$

$$\text{either} \quad S_n((j-1)h)u \quad \text{or} \quad S_n(jh) = S_n((j-1)h)d, \tag{6}$$

where $u = \mathrm{e}^{\mu h + \sigma\sqrt{h}}$ (which corresponds to $Y_j = 1$) and $d = \mathrm{e}^{\mu h - \sigma\sqrt{h}}$ (which corresponds to $Y_j = -1$).

As we have just seen in the previous section, the *real life probability* on the space of functions $S_n(t)$ (defined by (5) and (6)) **converges** to a probability corresponding to the Geometric Brownian motion.
What can we say about the convergence of the *risk-neutral probabilities* defined on the space of sequences (5) and thus also on the space of functions (6)?
The answer is given by the following statement.

5.1.2. Generalization of the B-S model

Theorem 5.1 ($S(t) \to (S_n(t)) \to \tilde{S}(t)$)

Suppose that $S(t) = S\mathrm{e}^{\mu t + \sigma W_t}$ and let r be the interest rate compounded continuously. Then the risk-neutral probability defined on the space of functions $S_n(t)$ converges to a risk neutral probability on the space of trajectories of the GBM given by

$$\tilde{S}(t) = S\mathrm{e}^{\tilde{\mu} t + \sigma W_t}, \quad \textit{where} \;\; \tilde{\mu} = r - \frac{1}{2}\sigma^2. \tag{7}$$

Definition 5.1

The process $\tilde{S}(t) = S\mathrm{e}^{\tilde{\mu} t + \sigma W_t}$ is called the *risk-neutral Geometric Brownian motion*.

Proof. Is not examinable.

5.1.2. Generalization of the B-S model

Before considering the first application of Theorem 5.1, let us recall the following general principle which allows one to compute the no-arbitrage price of a derivative in terms of the risk-neutral probability.

This theorem was discussed in the FMI course and should be known to you.
However, since it is a very important principle which we shall use also in the future, we shall prove it here - the proof is very short and simple.

5.1.2. Generalization of the B-S model

Theorem 5.2 ($C = e^{-rT}\tilde{E}$)

Suppose that

1. The payoff function of a derivative on a share is $R(T)$,

2. The payoff time is T.

3. The interest rate compounded continuously is r.

Then the price C of this derivative is

$$C = \mathrm{e}^{-rT}\tilde{\mathbb{E}}[R(T)], \tag{8}$$

where $\tilde{\mathbb{E}}$ is the expectation over the risk-neutral probability (RNP for short).

Please pay attention to the position of $\sim$.

5.1.2. Generalization of the B-S model

Proof.

If we buy the derivative for C then our return at time T is

$$R(T) - C\mathrm{e}^{rT}$$

(which is simply the difference between the payoff and our debt to the bank). The Arbitrage Theorem states that the expectation of the return over the RNP should be zero, that is

$$\tilde{\mathbb{E}}(R(T) - C\mathrm{e}^{rT}) = 0 \quad \text{or, equivalently,} \quad \tilde{\mathbb{E}}(R(T)) - C\mathrm{e}^{rT} = 0.$$

Hence

$$C = \mathrm{e}^{-rT}\tilde{\mathbb{E}}[R(T)].\square$$

0	1
Borrow from the bank C	$-Ce^{rT}$
to buy the derivative $-C$	$R(T)$

5.1.2. Generalization of the B-S model

The following statement is in fact a corollary of Theorems 5.1 and 5.2.

Theorem 5.3 ($C = e^{-rT} E(\sim)$)

Suppose that
1. The price of a share follows the GBM: $S(t) = S\mathrm{e}^{\mu t + \sigma W_t}$
2. The payoff function of a derivative on this share is $R(S(t_1), S(t_2), \cdots, S(t_k))$, *where* $0 \leq t_1 < t_2 < \ldots < t_k \leq T$.
3. The payoff time is T.
4. The interest rate compounded continuously is r.
Then the price C *of this derivative is*

$$C = \mathrm{e}^{-rT}\mathbb{E}[R(\tilde{S}(t_1), \tilde{S}(t_2), \cdots, \tilde{S}(t_k))], \tag{9}$$

where $\tilde{S}(t) = S\mathrm{e}^{\tilde{\mu}t + \sigma W_t}$, $\tilde{\mu} = r - \frac{1}{2}\sigma^2$.

Please pay attention to the position of $\sim$ in Equation 9 (move $\sim$ inside $E[\ ]$).

5.1.2. Generalization of the B-S model

Proof.
The payoff of our derivative at time T is

$$R(T) = R(S(t_1), S(t_2), \cdots, S(t_k)).$$

By Theorem 5.2,

$$C = \mathrm{e}^{-rT} \tilde{\mathbb{E}} \left[R(S(t_1), S(t_2), \cdots, S(t_k)) \right]. \quad (10)$$

where $\tilde{\mathbb{E}}$ is the expectation computed over the risk-neutral probability. By Theorem 5.1, the risk-neutral probability on the space of paths of the GBM is that corresponding to $\tilde{S}(t)$. This means that in order to compute the expectation in (10) over the risk-neutral probability we have to replace $S(t_1), S(t_2), \cdots, S(t_k)$ by $\tilde{S}(t_1), \tilde{S}(t_2), \cdots, \tilde{S}(t_k)$, that is:

$$\tilde{\mathbb{E}} \left[R(S(t_1), S(t_2), \cdots, S(t_k)) \right] = \mathbb{E} \left[R(\tilde{S}(t_1), \tilde{S}(t_2), \cdots, \tilde{S}(t_k)) \right].$$

This proves that

$$C = \mathrm{e}^{-rT} \mathbb{E}[R(\tilde{S}(t_1), \tilde{S}(t_2), \cdots, \tilde{S}(t_k))],$$

where $\tilde{S}(t) = S\mathrm{e}^{\tilde{\mu}t + \sigma W_t}$, $\tilde{\mu} = r - \frac{1}{2}\sigma^2$. □

5.2. Examples

The three example below show how to use the theorems mentioned above for the derivation of formulae two of which are known from the FMI course.

5.2.1. The European Call Option and the B-S Formula

The first example:
Recall the following notation: we write $\mathrm{Call}(K, T)$ instead of saying European call option with strike price K and expiration time T.

The payoff function for the $\mathrm{Call}(K, T)$ is $R(S(T)) = (S(T) - K)^+$.
This means that we should use Equation (9) with $k = 1$ and $t_1 = T$.
The price C of this option is therefore given by

$$C = \mathrm{e}^{-rT}\mathbb{E}(\tilde{S}(T) - K)^+. \tag{11}$$

5.2.1. The European Call Option and the B-S Formula

This formula is thus a very particular case of (9).
Yet (11) is the main part of the calculation which leads to the famous Black-Scholes formula:

$$C = C(S, T, K, \sigma, r) = S\Phi(\omega) - K\mathrm{e}^{-rT}\Phi(\omega - \sigma\sqrt{T}), \tag{12}$$

where

$$\omega = \frac{\ln \frac{S}{K} + rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}$$

and $\Phi(x)$ is the cumulative distribution function of a standard Normal random variable, that is

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-u^2/2}\,du\,.$$

5.2.1. The European Call Option and the B-S Formula

Remark.

In this module, the expression ‘the Black-Scholes Formula’ (BS formula for short) is usually associated with formula (12) (which is the most classical form of this formula). Also,in financial literature the name BS formula is often referring to more complicated versions of (12).

However, (12) is an easy corollary of another, and in fact more important form of the BS formula, namely (11).

Indeed, once (11) has been established, we can write it as an integral

$$C = \mathrm{e}^{-rT} \int_{-\infty}^{\infty} (S\mathrm{e}^{\tilde{\mu}T+\sigma\sqrt{T}x} - K)^{+} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} dx \qquad (13)$$

and the rest is a Calculus 2 exercise.

5.2.2. The European Put Option

The second example:
As you will know, the European put option $\mathrm{Put}(K, T)$ allows you to sell the underlying share for £K at the pre-negotiated time T (which is its expiry time).

To compute the gain, consider 2 cases.

5.2.2. The European Put Option

1. If $S(T) \geq K$ then your option is useless and you have to return to the bank £$C\mathrm{e}^{rT}$ which is the value at time T of the £C you have borrowed from the bank at time $t = 0$ to buy the option. Hence your **gain** is $-C\mathrm{e}^{rT}$.

   0 ——————————— 1

   Borrow from the bank C | $-Ce^{rT}$
   to buy the option $-C$ | 0

2. If $S(T) < K$, then (a) you buy the share for £$S(T)$ and sell it for £K (because you have the put option),
   (b) you return to the bank £$C\mathrm{e}^{rT}$ (as in the first case).
   Your **gain** thus is $(K - S(T)) - C\mathrm{e}^{rT}$.

   0 ——————————— 1

   Borrow from the bank C | (b) $-Ce^{rT}$
   to buy the option $-C$ | (a) buy the share $-S(T)$, and sell the share at K

5.2.2. The European Put Option

In other words, your **payoff** function is $R(S(T)) = (K - S(T))^+$ (and your **gain** is $(K - S(T))^+ - C\mathrm{e}^{rT}$).
So, by Equation 9, we obtain

$$C = \mathrm{e}^{-rT}\mathbb{E}(K - \tilde{S}(T))^+.$$

This expression can be transformed into a formula which is similar to the one obtained for the price of a European call option.
The calculations are similar to those for the $\mathrm{Call}(K, T)$.
However, the Call-Put parity formula provides a much shorter derivation.

5.2.3. The derivative with payoff $R(T) = \frac{1}{T}\int_0^T S(t)\mathrm{d}t$

The third example:
Suppose that the price of an asset is driven by a GBM: $S(t) = S\mathrm{e}^{\mu t+\sigma W_t}$. Consider a derivative on this asset with a payoff function

$$R(T) = \frac{1}{T}\int_0^T S(t)\mathrm{d}t.$$

What is the risk-neutral price C of this derivative?
Note the particularity of this example: the payoff function depends on all values of $S(t)$, $t \in [0,\ T]$.

5.2.3. The derivative with payoff $R(T) = \frac{1}{T}\int_0^T S(t)\mathrm{d}t$

The answer follows from Theorems 5.2 and 5.3. Namely, by Theorem 5.2 and by the definition of $R(T)$,

$$C = \mathrm{e}^{-rT}\tilde{\mathbb{E}}\left(\frac{1}{T}\int_0^T S(t)\mathrm{d}t\right) = \frac{\mathrm{e}^{-rT}}{T}\tilde{\mathbb{E}}\left(\int_0^T S(t)\mathrm{d}t\right). \tag{14}$$

By Theorem 5.3,

$$\tilde{\mathbb{E}}\left(\int_0^T S(t)\mathrm{d}t\right) = \mathbb{E}\left(\int_0^T \tilde{S}(t)\mathrm{d}t\right).$$

The remarkable fact is that it is possible to change the order of the two operations:

$$\mathbb{E}\left(\int_0^T \tilde{S}(t)\mathrm{d}t\right) = \int_0^T \mathbb{E}\left(\tilde{S}(t)\right)\mathrm{d}t.$$

In other words, rather than first computing the integral and then the expectation, we can first compute the expectation and after that compute the integral.
We shall use this fact now and also later in the course but its proof is beyond our means.

5.2.3. The derivative with payoff $R(T) = \frac{1}{T}\int_0^T S(t)\mathrm{d}t$

The rest is simple because we know (see Week 1) that

$$\mathbb{E}\left(\tilde{S}(t)\right) = \mathbb{E}\left(S\mathrm{e}^{\tilde{\mu}t+\sigma W_t}\right) \overset{(*)}{=} S\mathrm{e}^{\tilde{\mu}t+\frac{\sigma^2}{2}t}.$$

Since $\tilde{\mu} + \frac{\sigma^2}{2} = r$ (Theorem 5.1), we have

$$\mathbb{E}\left(\tilde{S}(t)\right) = S\mathrm{e}^{rt}$$

and we obtain

$$\mathbb{E}\left(\int_0^T \tilde{S}(t)\mathrm{d}t\right) = \int_0^T S\mathrm{e}^{rt}\mathrm{d}t = \frac{S}{r}(\mathrm{e}^{rT} - 1).$$

Finally we obtain from (14):

$$C = \frac{\mathrm{e}^{-rT}S}{rT}(\mathrm{e}^{rT} - 1) = \frac{S}{rT}(1 - \mathrm{e}^{-rT}).$$

5.2.3. The derivative with payoff $R(T) = \frac{1}{T}\int_0^T S(t)\mathrm{d}t$

(*) is because of Theorem 1.1.

Theorem 1.1 (revisit):
If $S(t)$ is a Geometric Brownian Motion with drift μ and volatility σ then

$$\mathbb{E}\left(S(t)\right) = S(0)\mathrm{e}^{\mu t + \frac{\sigma^2 t}{2}}.$$

6. The Greeks

6. The Greeks

In this section we shall study the dependence of the Black-Scholes price

$$C = C(S, T, K, \sigma, r) = \mathrm{e}^{-rT}\mathbb{E}((S(T) - K)^+)$$

on the parameters S, T, K, σ, and r.

More precisely, we shall compute the **first order partial derivatives** of C with respect to each of the parameters.
The value of each such derivative shows the **sensitivity** of the price to small changes in the parameter.
And if we know just that a derivative is positive (negative) then we also know that C is growing (decaying) when the parameter is growing.

6. The Greeks

Let us introduce the names for these derivatives. First of all they have a collective name - they are called **the Greeks**. The reason for this term is that these derivatives are denoted by Greek letters. Here are the most common of them:

$$\Delta = \frac{\partial C}{\partial S} \quad \text{is called delta}$$

$$\nu = \frac{\partial C}{\partial \sigma} \quad \text{is called vega (see remark below)} \ \to \text{volatility}$$

$$\rho = \frac{\partial C}{\partial r} \quad \text{is called rho} \quad \to \text{return}$$

$$\theta = \frac{\partial C}{\partial T} \quad \text{is called theta} \quad \to \text{time}$$

6. The Greeks

Remark.
Vega is not the name of any Greek letter. The letter we actually use is called nu. Nevertheless, this is how this derivative is called in financial literature - presumably because ν looks similar to the Latin $\mathcal{V}$ (vee).
The very fact that these names were invented and are widely used shows how important these derivatives are.

6. The Greeks

You are not required to remember the calculations presented below.
I do however strongly suggest that you understand them.

6. The Greeks

In what follows, we shall use the Black & Scholes formula for the price of the call option given by

$$C = C(S, T, K, \sigma, r) = S\Phi(\omega) - K\mathrm{e}^{-rT}\Phi(\omega - \sigma\sqrt{T})$$

where

$$\omega = \frac{\ln \frac{S}{K} + rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T} \qquad \text{and} \qquad \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-u^2/2}\, du\,. \tag{15}$$

Observe that the price C of the call option does NOT depend on the drift parameter μ of the GBM which describes the behaviour of the price $S(T)$ of the underlying share.

6. The Greeks

We shall show that $C = C(S, T, K, \sigma, r)$ is a decreasing function of K (while all other parameters remain constant) and it is an increasing function of all other parameters, that is of S, T, σ, r. These properties follow from the following **lemma**:

The partial derivatives of $C = C(S, T, K, \sigma, r) = \mathrm{e}^{-rT}\mathbb{E}((S(T) - K)^+)$ are:

$$\frac{\partial C}{\partial K} = -\mathrm{e}^{-rT}\Phi(\omega - \sigma\sqrt{T})$$

$$\frac{\partial C}{\partial S} = \Phi(\omega)$$

$$\frac{\partial C}{\partial r} = KT\mathrm{e}^{-rT}\Phi(\omega - \sigma\sqrt{T})$$

$$\frac{\partial C}{\partial \sigma} = S\sqrt{T}\Phi'(\omega)$$

$$\frac{\partial C}{\partial T} = \frac{\sigma}{2\sqrt{T}}S\Phi'(\omega) + Kr\mathrm{e}^{-rT}\Phi(\omega - \sigma\sqrt{T}).$$

6. The Greeks

The above formulae show that $\frac{\partial C}{\partial K} < 0$ and all other derivatives are positive, namely $\frac{\partial C}{\partial S} > 0$, $\frac{\partial C}{\partial r} > 0$, $\frac{\partial C}{\partial \sigma} > 0$, $\frac{\partial C}{\partial T} > 0$. Hence C is indeed a decreasing function in K and increasing function in all other parameters.