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MTH5126 Statistics for Insurance

Dr Lei Fang
lei.fang@qmul.ac.uk

Week 2

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Reinsurance

- The claims on an insurance company have to be paid in full.
- However, to protect itself from large claims, the company may itself take out an insurance policy.
- The terms of such a policy are covered by a document called a **reinsurance treaty**.
- This form of insurance is called reinsurance and there are specialist reinsurance companies (such as Swiss Re, RGA, Munich Re, SCOR and many others).

There are two main types of reinsurance:

1. **Proportional** reinsurance.
2. **Non-Proportional** reinsurance.

Proportional Reinsurance

- Here, the cedant / ceding company /direct writer and the reinsurer share in all aspects of the risk. Most importantly, this means they share in the cost of claims for the risk.
- The cedant must pay a premium for this. The premium calculation basis will be specified in the reinsurance treaty.

For example, for a property that has been insured against fire, the cedant (insurance company) might retain 80% of the premium and then be liable to pay the same 80% proportion for all claims made - regardless of the size of claim.

There are two forms of proportional reinsurance:

1. **Quota share reinsurance:** The proportions are the same for all risks. In this module, we focus on this type of proportional reinsurance.
2. **Surplus reinsurance:** The proportions can vary from one risk to the next.

Non-Proportional Reinsurance

- Here the cedant pays a fixed premium to the reinsurer.
- The reinsurer will then pay the part of the claim that lies in a particular reinsurance ***layer***, e.g., claim amounts between £4m and £5m.
- The layer will be defined by a lower limit *the retention limit* (in this case £4m) and an upper limit.
- The upper limit can simply be infinite if cover is *unlimited*.

There are 2 forms of non-proportional reinsurance.

1. **Individual Excess of Loss (XOL)**
2. **Stop Loss**

Non-Proportional Reinsurance

1. Individual Excess of Loss (XOL)

The reinsurer will make the payment when the amount of a particular claim exceeds a specified ***excess point*** or ***retention***.

For example, the reinsurer might pay the excess when a claim from a car insurance policy exceeds £50,000 but to an upper limit of £1m. In this module, we focus on this form of non-proportional reinsurance.

2. Stop Loss

The reinsurer will make the payment if the total claim amount for a specified group of policies exceeds a specified amount.

This amount is usually expressed as a percentage of the premium. For example, a reinsurer might agree to pay 85% of the excess when the total claim amount for all car insurance policies exceeds 100% of the total premium, with no upper limit.

Reinsurance

- **Gross claim amount:** The original values without any adjustments for reinsurance.
- **Gross premium income :** The original values without any adjustments for reinsurance.
- **Net claim amount:** The actual claim amount that the cedant ends up paying after allowing for payments under the reinsurance arrangements.
- **Net premium income:** The actual premium amount that they get to keep after making any payments for reinsurance.

Notation:

- X is the gross claim amount r.v.
- Y is the net claim amount r.v.
- Z is the amount paid by the reinsurer in respect of a single claim.

What is the relationship between X, Y and Z?

Answer: $X = Y + Z$

Excess of loss reinsurance

The insurer will pay any claim in full to an amount *M*, the retention level. Any amount above *M* will be borne by the reinsurer. If the claim is amount *X*, then the insurer pays *Y* where:

$$Y = X, \qquad \text{if } X \leq M$$

$$Y = M, \qquad \text{if } X > M$$

The reinsurer pays the amount $Z = X - Y$, where

$$Z = 0, \qquad \text{if } X \leq M$$

$$Z = X - M, \qquad \text{if } X > M$$

The insurer's liability is affected in two ways by this type of reinsurance:

1. The mean amount paid is reduced.
2. The variance of the amount paid is reduced.

Because excess of loss reinsurance puts an upper limit on large claims.

Excess of loss reinsurance

Insurer's perspective

- The mean amount paid by the insurer without reinsurance is:

  $E(X) = \int_0^\infty xf(x)dx$

  where $f(x)$ is the PDF of the claim amount X.

- With a retention level of *M,* the mean amount paid by the insurer becomes

  $E(Y) = \int_0^M xf(x)dx + MP(X > M)$

  (Since $E(Y) = \int_0^M xf(x)dx + \int_M^\infty Mf(x)dx$ from the definition of Y.)

- More generally, the moment generating function of Y, the amount paid by the insurer is:

  $$M_Y(t) = E(e^{tY}) = \int_0^M e^{tx}f(x)dx + e^{tM}P(X > M)$$

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Excess of loss reinsurance

Reinsurer's perspective

Under excess of loss the reinsurer will pay Z, were

$$Z = \begin{cases} 0 & \text{if } X \le M \\ X - M & \text{if } X > M \end{cases}$$

So, with a retention level of *M* the mean amount paid by the reinsurer is then:

$$E(Z) = \int_M^\infty (x - M) f(x) dx$$

More generally, the moment generating function of Z, the amount paid by the reinsurer is:

$$M_Z(t) = E(e^{tZ}) = \int_0^M e^{t0} f(x)dx + \int_M^\infty e^{t(x-M)} f(x)dx$$

The reinsurer may only have a record of claims that are greater than *M*. The reinsurer therefore has the problem of estimating the underlying claims distribution when only those claims greater than *M* are observed. We say that the reinsurer is observing claims from a ***truncated distribution***.

Excess of loss reinsurance

Reinsurer's perspective

Let W be the random variable with this truncated distribution. Then

$$W = X - M, \quad X > M.$$

Suppose that the reinsurer is only informed of claims greater than the retention limit M and has a record of $W = X - M$.

Question: What is g(W), the PDF of the amount W paid by the reinsurer if f(X) and F(X) are PDF and CDF of the underlying claim amounts, respectively?

Answer:

$$\mathbf{G(W)}=P(W < w) = P(X < w + M | X > M)$$

$$= \frac{P(X < w + M \text{ and } X > M)}{P(X > M)}$$

(using Bayes' Theorem)

$$= \frac{P(M < X < w + M)}{P(X > M)}$$

$$= \int_M^{w+M} \frac{f(x)}{1 - F(M)} dx$$

(Since F(M) = P(X < M))

$$= \frac{F(w + M) - F(M)}{1 - F(M)}$$

This uses the standard result that:
$P(a < X < b) = \int_a^b f(x)dx = F(b) - F(a)$

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Excess of loss reinsurance

Reinsurer's perspective

Finally, differentiating with respect to w we get that the PDF of the reinsurer's claims is:

$$g(w) = \frac{f(w+M)}{1-F(M)}, w > 0$$

Excess of loss reinsurance

Example: Mean amount paid by reinsurer when X~N(500,400)

Claims from a portfolio have a $N(500, 400)$ distribution and there is a retention limit of M = 550. Find the mean amount paid by the reinsurer on all claims.

Answer:

$$E(Z) = \int_0^{550} 0 \times f_X(x)dx + \int_{550}^{\infty} (x - 550) f_X(x)dx$$

where $f_X(x)$ is the PDF of the $N(500, 400)$ distribution.

The PDF of a normal distribution is $\frac{1}{\sqrt{2\pi\sigma^2}} \exp[\frac{-(x-\mu)^2}{2\sigma^2}]$

Substituting in $u = \frac{x-500}{20}$ in the second integral, we get:

$$E(Z) = \int_{2.5}^{\infty} (20u - 50) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

$$= 20 \left[-\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} \right]_{2.5}^{\infty} - 50[1 - \Phi(2.5)] \quad = 0.35 – 0.31 = 0.04$$

Proportional reinsurance

In proportional reinsurance the insurer pays a fixed proportion of the claim, whatever the size of the claim. If a claim is for an amount X, then the insurance company will pay Y where

$$Y = \alpha X, \qquad 0 < \alpha < 1.$$

The parameter α is known as *the* **retained proportion** or **retention level**.

Note that the term retention level is used in both excess of loss and proportional reinsurance even though it means different things.

The distribution of both amounts can be found by a simple change of variable:

$$Y = \alpha X,$$

$$Z = (1 - \alpha)X.$$

Proportional reinsurance

Example: Distribution of claims paid by reinsurer when X ~ Gamma(α, λ)

Question:

A sample of a reinsurer's payments made under a proportional reinsurance arrangement consists of the following values, in (£ 000s):

4.6, 6.8, 22.9, 1.4, 3.8, 10.2, 19.4, 32.1

If the original claim amounts are modelled using a $Gamma(\alpha, \lambda)$ distribution and the retained proportion is 80%, find the distribution of the reinsurer's claims and hence estimate the parameters α and λ using a method of moments approach.

Answer:

- If the original claim payments X are $Gamma(\alpha, \lambda)$, then the reinsurer's payments have the distribution

$$Y = 0.2X.$$

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Proportional reinsurance

Example: Distribution of claims paid by reinsurer when X ~ Gamma(α, λ)

Answer (continued):

Substitute y = 0.2x in the integral of the PDF:

$$\int_0^\infty \frac{1}{\Gamma(\alpha)} \lambda^\alpha x^{\alpha-1} e^{-\lambda x} dx$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha)} \lambda^\alpha (5y)^{\alpha-1} e^{-5\lambda y} 5dy$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha)} (5\lambda)^\alpha y^{\alpha-1} e^{-5\lambda y} dy$$

This is the PDF of a $Gamma(\alpha, 5\lambda)$ distribution, which has a mean of $\frac{\alpha}{5\lambda}$ and a variance of $\frac{\alpha}{25\lambda^2}$

Proportional reinsurance

Example: Distribution of claims paid by reinsurer when X ~ Gamma(α,λ)

Answer (continued): The sample mean and variance (using denominator *n*) are:

$$\bar{x} = \frac{\sum x_i}{8} = \frac{101.2}{8} = 12.65$$

and $s^2 = \frac{\sum x_i^2}{8} - \bar{x}^2 = \frac{2119.02}{8} - 12.65^2 = 104.855$

So the equations for α and λ are:

$$\frac{\alpha}{5\lambda} = 12.65$$

and

$$\frac{\alpha}{25\lambda^2} = 104.855$$

Solving these gives $\alpha = 1.526$ and $\lambda = 0.02413$

Useful integral formulae *Lognormal distribution*

There are some useful integral formulae that simplify reinsurance calculations when working with some distributions.

If $f_X(x)$ is the PDF of the $LogNormal(\mu, \sigma^2)$ distribution, then:

$$\int_L^U x^k f_X(x)dx = e^{k\mu+\frac{1}{2}k^2\sigma^2}\left[\Phi(U_k) - \Phi(L_k)\right]$$

Where :

$$L_k = \frac{logL - \mu}{\sigma} - k\sigma$$

and

$$U_k = \frac{logU - \mu}{\sigma} - k\sigma$$

and: $\Phi(z)$ is the distribution function of the standard normal distribution.

Useful integral formulae *Lognormal distribution*

Proof:

Using the formula for $f_X(x)$ and the substitution $t = \frac{\log x - \mu}{\sigma} - k\sigma$ gives:

$$\int_L^U x^k f_X(x)dx = \int_L^U x^k \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\log x-\mu}{\sigma}\right)^2} dx$$

$$= \int_{L_k}^{U_k} e^{k(\mu+\sigma t+k\sigma^2)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(t+k\sigma)^2} dt$$

$$= \int_{L_k}^{U_k} \frac{1}{\sqrt{2\pi}} e^{k\mu+k\sigma t+k^2\sigma^2} e^{-\frac{1}{2}t^2-k\sigma t-\frac{1}{2}k^2\sigma^2} dt$$

$$= e^{k\mu+\frac{1}{2}k^2\sigma^2} \int_{L_k}^{U_k} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$$

$$= e^{k\mu+\frac{1}{2}k^2\sigma^2} [\Phi(U_k) - \Phi(L_k)]$$

When $L = 0$ or $U = \infty$, these formulae can be simplified using the facts that:

$$\Phi(-\infty) = 0$$

$$\Phi(0) = \frac{1}{2}$$

$$\Phi(\infty) = 1$$

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Useful integral formulae *Normal distribution*

A similar formula can be derived for the first moment of the normal distribution.

If $f_X(x)$ is the PDF of the $N(\mu, \sigma^2)$ distribution then:

$$\int_L^U x f_X(x) dx = \mu \left[\Phi(U') - \Phi(L')\right] - \sigma \left[\phi(U') - \phi(L')\right]$$

Where

$$L' = \frac{L - \mu}{\sigma}$$

and

$$U' = \frac{U - \mu}{\sigma}$$

With $\phi(z)$ and $\Phi(z)$ being the probability density function and distribution function respectively of the standard normal distribution.

The proof left as a worksheet exercise.

Inflation

- The examples that we have looked at so far have **assumed** that **claim distributions remain constant** over time. (Or at least a sufficiently short time period for constancy to be a valid assumption).
- In reality, claims are likely to increase due to inflation - at least in the longer term.
- A claims distribution that is suitable for describing claim amounts in one year may well not be suitable a year or two later.
- With excess of loss arrangements, inflation can cause an issue.

Inflation

Question: Suppose that the claims X are inflated by a factor of k, but the retention, M remains fixed. What effect does this have on the **excess of loss** reinsurance arrangement?

Answer: The amount claimed is kX and the amount paid by the insurer, Y, is:

$$Y = kX, \quad \text{if } kX \leq M$$

$$Y = M, \quad \text{if } kX > M$$

- The mean amount paid by the insurer is:

$$E(Y) = \int_0^{\frac{M}{k}} kxf(x)dx + MPr(X > \frac{M}{k})$$

One important general point that can be made is that the new mean claim amount paid by the insurer is **not** k times the mean claim amount paid by the insurer without inflation.

- A similar approach can be taken in situations where the retention limit is linked to an index of inflation.

Estimation when the sample is censored

Consider the problem of estimation in the presence of excess of loss reinsurance:

- Our claims record might only show the net claims paid by the insurer. A typical claims record might be: x_1, x_2, M, x_3, M, x_4, x_5
- We require from this an estimate of the underlying gross claims distribution.
- We can't use the method of moments because we can't even compute the mean claim amount.
- We may be able to use the method of percentiles depending on the retention level, M. If it is high enough then maybe only the higher sample percentiles were affected by the reinsurance.

Data in the form here is said to be **censored**. In general, a censored sample occurs when some values are recorded exactly and the remaining values are known to exceed a particular value, in this instance the retention limit M.

- The best method to use for estimation from censored samples is maximum likelihood.

Estimation when the sample is censored

- We can apply maximum likelihood by splitting the likelihood function into two parts:

If the values of $x_1, x_2, \ldots, x_n$ are recorded exactly then these contribute a factor of

$$L_1(\underline{\theta}) = \prod_1^n f(x_i; \underline{\theta})$$

If a further m claims are referred to the reinsurer, then the insurer records a payment of M for each of these claims. These censored values then contribute a factor of

$$L_2(\underline{\theta}) = \prod_1^m P(X > M)$$

The complete likelihood function is then

$$L(\underline{\theta}) = \prod_1^n f(x_i; \underline{\theta}) \times [1 - F(M; \theta)]^m$$

where $F(M; \underline{\theta})$ is the CDF of the claims distribution.

Estimation when the sample is censored

- The reason for multiplication is that the likelihood reflects the probability of getting the n claims with known values *and* m claims exceeding M.
- We are also assuming here that the **claims are independent**.

Policy excess

Insurance policies with an excess are common in motor insurance and many other kinds of property and accident insurance. Under this kind of policy, **the insured agrees to carry the full burden of the loss up to a limit L, called the excess.**

- If the loss is an amount, X, greater than L, then the policyholder will claim only $X - L$.
- If Y is the amount actually paid by the insurer, then:

$$Y = 0, \quad \text{if } X \leq L$$

$$Y = X - L, \quad \text{if } X > L$$

- **Question:** Is the premium due on a policy with an excess greater than or less than that on a policy without an excess?

  Answer: Less.

- The position of the insurer for a policy with an excess is exactly the same as that of the reinsurer under excess of loss reinsurance.
- The position of the policyholder as far as losses are concerned is exactly the same as that of an insurer with an excess of loss reinsurance contract.

Risk Models

Introduction

We begin by describing the main features of a general insurance policy.

- Construct models appropriate for short term insurance contracts in terms of the numbers of claims, N, and the amounts of individual claims, X.
- Describe the major simplifying assumptions underlying the models above.
- Introduce the idea of a *compound distribution*.
- Define and use the compound Poisson and compound binomial distributions.
- Look at the mean, variance, coefficient of skewness of the compound Poisson and compound binomial distributions.
- Repeat the above for both the insurer and reinsurer after the operation of simple forms of proportional and excess of loss reinsurance.

Features of a general insurance product

Insurable risk

Generally, for a risk to be insurable:

1. The policyholder must have an interest in the risk being insured, to distinguish between insurance and a wager.

2. A risk must be of a financial and reasonably quantifiable nature.

Features of a general insurance product

Ideally, risk events also need to meet the following criteria if they are to be insurable:

- Individual risk events should be independent of each other.
- The probability of the event should be relatively small. In other words, an event that is nearly certain to occur is not conducive to insurance.
- Large numbers of potentially similar risks should be pooled in order to reduce the variance and hence achieve more certainty.
- There should be an ultimate limit on the liability undertaken by the insurer.
- Moral hazards should be eliminated as far as possible because these are difficult to quantify, result in selection against the insurer and lead to unfairness in treatment between one policyholder and another.

Features of a general insurance product

- Cover is normally for a fixed period, most commonly one year, after which it has to be renegotiated.
- There is normally no obligation on either the insurer or the insured to continue the arrangement after this, although in practice a need for continuing cover is assumed to exist.
- Claims are not of fixed amounts and the amount of a loss as well as the fact needs to be proved before a claim can be settled.
- A claim occurring does not bring a policy to an end.
- Claims may occur at any time during the policy period. Although there is normally a contractual obligation on the policyholder to report a claim to the insurer as quickly as possible, notification may take some time if the loss is not evident immediately.
- Settlement of the claim may also take a long time, especially if protracted legal proceedings are needed or if it is not straightforward to determine the extent of the loss.

Features of a general insurance product

- However, from the moment of the event giving rise to the claim the ultimate settlement amount is a liability of the insurer. Estimating the amounts of money that need to be reserved to settle these liabilities is one of the most important areas of actuarial involvement in general insurance.
- Note that classes of insurance in which claims tend to take a short time to settle are known as short-tail and vice-versa, although the dividing line between the two categories is not always distinct.
- The range of general insurance products is very wide and constantly changing.

For those of you continuing your studies to become an actuary after your undergraduate studies, you will meet these in subject CP1 - Actuarial Practice.

Models for short term insurance contracts

The basic model

- Many forms of non-life insurance, for example motor insurance, can be regarded as short-term contracts.
- We can also regard as short-term some forms of life insurance, for example group life insurance contracts and one-year term assurances.
- A short-term insurance contract can be defined as having the following attributes:
  1) The policy lasts for a fixed, and relatively short, period of time, typically one year.
  2) The insurance company receives a premium from the policyholder.
  3) In return, the insurer pays claims that arise during the term of the policy.
- At the end of the policy term the policyholder may or may not renew the policy.
- If it is renewed, then the premium payable may not be the same as the previous period of cover.
- The insurer may choose to pass part of the premium to a reinsurer.
- In return, the reinsurer will reimburse the insurer part of the cost of the claims during the policy's term according to some agreed formula.

Models for short term insurance contracts

The basic model

- An important feature of a short-term insurance contract is that the premium is set at a level to cover claims arising during the (short) term of the policy only.
- This contrasts with life assurance policies where mortality rates increasing with age mean that the (level) annual premium in the early years would be more than sufficient to cover the expected claims in those years. The excess amount would then be accumulated as a reserve to be used in the later years when the premium on its own would be insufficient to meet the cost of expected cost of claims.
- A risk includes either a single policy or a specified group of policies. For ease of terminology, we will assume that the term of the policy is one year, but it could equally well be any other short period.

Models for short term insurance contracts

The basic model

- Let the random variable S denote the aggregate claims paid by the insurer in the year in respect of this risk.
- We will construct models for this random variable S.
- We start with looking at *collective* risk models and later we will look at *individual* risk models.
- A first step in the construction of a collective risk model is to write S in terms of the number of claims arising in the year, denoted by the random variable N and the amount of each individual claim.
- Let the random variable X_i denote the amount of the i-th claim. Then:

$$S = \sum_{i=1}^{N} X_i$$

where the summation is taken to be zero if N is zero.

Models for short term insurance contracts

The basic model

This decomposition of S allows consideration of claim numbers and claim amounts separately. A practical advantage of this is that factors affecting claim numbers and claim amounts may well be different.

Take motor insurance as an **example**:

- a prolonged spell of bad weather may have a significant effect on claim numbers but little or no effect on the distribution of the individual claim amounts.
- On the other hand, inflation may have a significant effect on the cost of repairing cars and hence on the distribution of individual claim amounts, but little or no effect on claim numbers.

➢ This approach is referred to as a **collective risk model** because it is considering the claims arising from a group of policies taken as a whole, rather than by considering the claims arising from each individual policy.

Models for short term insurance contracts

Compound distribution

- The random variable S is the sum of a random number of random quantities and is said to have a *compound distribution*.

$$S = \sum_{i=1}^{N} X_i$$

- Because compound distributions arise commonly in general insurance, the random variable N is often referred to as "the number of claims" and the distribution of the random variables $X_1, X_2, \ldots$ is referred to as the "individual claim size distribution", even where the compound distribution arises in another context.

- To define a compound distribution, you need to know two components:
  1. The distribution of N (which is a *discrete* distribution)
  2. The distribution of the X_i's (which may be *any* distribution)

Models for short term insurance contracts

Compound distribution

- Note that, if the distribution of the $\boldsymbol{X_i}$**'s is continuous**, then S will have a *mixed distribution* meaning **partly discrete and partly continuous**. This is because of the possibility that $N = 0$.
- The problems we will look at are the derivation of the moments and distribution of S in terms of the moments and distributions of N and the X_i's.
- Both will be studied with and without simple forms of reinsurance.
- The corresponding problems for the reinsurer will also be studied, *i.e.,* the derivation of the moments and distribution of the aggregate claims paid in the year in respect of this risk by the reinsurer.

Models for short term insurance contracts

Simplifications in the basic model

The model for short-term insurance described contains several simplifications as compared to a real insurance company.

- The first of these is that it is usually assumed that the moments and sometimes the distributions of N and the X_i's are **known with certainty**. In practice these would probably be estimated from some relevant data using methods already studied.

For example, we might assume that claim amounts have a $Gamma(500, 5)$ distribution. However, in practice it might not be possible to make such simple assumptions:

1. There may not be an appropriate theoretical **distribution** that models the distribution of claim amounts actually paid sufficiently well.
2. Even if the shape of the distribution is satisfactory, appropriate **parameter** values may change over time, even in the short term.
3. There may not be sufficient **homogeneity** in the portfolio. For example, different policies may produce claim amounts that have different sizes.

Models for short term insurance contracts

Simplifications in the basic model

- Another simplification is to assume, at least implicitly, that claims are **settled** more or less **as soon as the incident** causing the claim **occurs**, so that, for example, the insurer's profit is known at the end of the year.
    1. In practice, there will be at least a short **delay** in the settlement of claims and in some cases the delay can amount to many years.
    2. This will be especially true when the extent of the loss is **difficult to determine**, for example if it is to be decided in a court of law.
    3. Delays will often lead to **inflationary pressure** leading to higher payments.

Models for short term insurance contracts

Simplifications in the basic model

- The model does not in general include any mention of **expenses**.

The premium is assumed to pay the claims and include a loading for profit. In practice, the premium paid by the policyholder(s) will also include a loading for expenses. It is possible to include expenses in the model in a very simple way.

1. The simplest way to allow for expenses would be to use a claim size distribution that was **artificially inflated** to allow for some sort of claim expense amount (e.g., by adding 20%).
2. Alternatively, we might express the random variable X as the sum of two other **random variables**, one to represent the actual claim amount and the other to represent the corresponding claim expense.

Models for short term insurance contracts

Simplifications in the basic model

- An important element in models for long-term insurance is a **rate of interest**.

This is because the excess premium income would be invested to build up reserves.

  - Interest is a relatively less important, but still important, feature of **short-term** insurance. It is possible to include interest in models for short-term insurance, but it is more usual to ignore it in elementary modelling (which is what we will do).

- There are a few **additional elements** included when setting the premium to be charged to policyholders, including the policyholder's **previous claims record**. These are all covered in CP1 - Actuarial Practice for those that go on to study this.
- The allowance for policyholder's claims experience could be based on claim frequency or claim severity (amount) but we will not look at that here.

Models for short term insurance contracts

Notation and assumptions

➢ Throughout this part of the course the following two important assumptions will be made:

- The random variables $\{X_i\}_{i=1}^{N}$ are independent and indentically distributed.
- The random variable N is independent of $\{X_i\}_{i=1}^{N}$

In words these assumptions mean that:

1. The **number** of claims is not affected by the **amount** of individual claims.
2. The **amount of a given individual** claim is not affected by the amount of any other individual claim.
3. The **distribution** of the amounts of individual claims does **not change** over the (short) term of the policy.

Models for short term insurance contracts

Notation and assumptions

- Throughout this part of the course, we will also assume that all claims are for **non-negative amounts**. This means that $P(X_i \leq x) = 0$ for $x < 0$.
- Many of the formulae in this part will be derived using the moment generating functions (MGF's) of S, N and X_i. These MGF's will be denoted by $M_S(t)$, $M_N(t)$ and $M_X(t)$ respectively and will be assumed to exist for some positive values of the dummy variable t.
- The **existence** of the MGF of a non-negative random variable for positive values of t **cannot generally be taken for granted**. For example, the MGF's of the Pareto and of the lognormal distributions do not exist for any positive value of t.
- However, all the formulae derived here with the help of MGF's can be derived, although less easily, without assuming the MGF's exist for positive values of t.

Models for short term insurance contracts

Notation and assumptions

- $G(x)$ and $F(x)$ shall denote the distribution functions of S and X_i respectively so that:

$$G(x) = P(S \leq x), \qquad F(x) = P(X_i \leq x)$$

- For convenience it will often be assumed that the density of $F(x)$ exists, and it will be denoted by $f(x)$.
- In cases where this density does not exist, so that X_i has a **discrete** or a **mixed** distribution, expressions such as:

$$\int_0^\infty x f(x) dx$$

  should be interpreted appropriately. This meaning should always be clear from the context.
- The k-th moment, $(k = 1, 2, 3, \ldots)$ of X_i about zero will be denoted by m_k.

Queen Mary
University of London

Models for short term insurance contracts

Notation and assumptions

Question: Write down formulae in terms of m_k for the mean, variance and coefficient of skewness of a random variable X_i, representing individual claim sizes.

Answer:

- The mean is: $E(X) = m_1$
- The variance is: $var(X) = m_2 - m_1^2$
- The coefficient of skewness is:

$$\frac{skew(X)}{[var(X)]^{\frac{3}{2}}}$$

$$= \frac{m_3 - 3m_2m_1 + 2m_1^3}{[m_2 - m_1^2]^{\frac{3}{2}}}$$