MTH5126 - Statistics for Insurance

Academic Year: 2022-23 **Semester: B**

Worksheet 9

Q1. Ruin Theory

An insurer calculates the annual premiums for fire insurance of flats by increasing the risk premium by 30% and adding a £30 loading.

The claim frequency is 3% and individual claim amounts can be assumed to be:
£2,000 with probability 0.9
£15,000 with probability 0.1

Calculate the insurer's adjustment coefficient for these policies, to 2 significant figures.

The adjustment coefficient is the unique positive solution to the following equation:

$$\lambda + cR = \lambda E\,(e^{RX})$$

The average claim size is:

$$E\,(X) = 0.9 \times 2{,}000 + 0.1 \times 15{,}000 = £3{,}300$$

So, the annual risk premium, E(S), is:

$$\lambda E(X) = 0.03 \times 3{,}300 = £99$$

So, the annual premium is:

$$c = (1+\theta)\lambda E(X) + 30 = 1.3 \times 99 + 30 = £158.70$$

So, the adjustment coefficient equation is:

$$0.03 + 158.70R = 0.03(0.9e^{2{,}000R} + 0.1e^{15{,}000R})$$

The equation for R has to be solved numerically this time (unlike the questions in previous worksheet).

Dividing by 0.03,

$$1 + 5290R = 0.9e^{2{,}000R} + 0.1e^{15{,}000R}$$

Writing $\tilde{R} = 1{,}000R$ gives:

$$1 + 5.29\tilde{R} = 0.9\,e^{2\tilde{R}} + 0.1e^{15\tilde{R}}$$

Expanding the RHS as a series to get a first approximation:

$$1+5.29\tilde{R} = 0.9(1+2\tilde{R}+(2\tilde{R})^2/2! +\ldots)+0.1(1+15\tilde{R}+(15\tilde{R})^2/2! +\ldots)$$

$$= 0.9 + 1.8\tilde{R} + 1.8\tilde{R}^2 + \ldots + 0.1 + 1.5\tilde{R} + 11.25\tilde{R}^2 + \ldots$$

$$= 1+3.3\tilde{R}+13.05\,\tilde{R}^2+\ldots$$

Ignoring higher order terms,

$$13.05\,\tilde{R}^2 - 1.99\,\tilde{R} = 0$$

$$\tilde{R}(13.05\,\tilde{R} - 1.99) = 0$$

$\tilde{R}$ is positive, cannot be 0. So

$$\tilde{R} \approx \frac{1.99}{13.05} = 0.1525$$

Evaluating

$$f(\tilde{R}) = 0.9\,e^{2\tilde{R}} + 0.1e^{15\tilde{R}} - 1 - 5.29\tilde{R}$$

for different values of $\tilde{R}$, we get:

$\tilde{R}$	$f(\tilde{R})$
0.15	0.37
0.1	0.018
0.09	-0.013
0.094	-0.0015
0.095	0.00156
0.0945	0.000011

So $\tilde{R}$ lies between 0.0094 and 0.00945.

So the value of the adjustment coefficient, R, is 0.0000094 (correct to 2 significant figures).

Q2. Ruin Theory

Show that the adjustment coefficient for a compound Poisson claims process satisfies the inequality:

$$R < \frac{2[\frac{c}{\lambda} - E(X)]}{E(X^2)}$$

and define what each of the symbols represents.

The adjustment coefficient, R, is the unique positive solution of the equation:

$$\lambda + cR = \lambda E\,(e^{RX})$$

The expected claim frequency, λ, is the expected number of claims occurring per unit of time.

The premium rate c is the constant amount of premium received per unit of time.

The claim size X is a random variable representing the amount of an individual claim.

Expanding the RHS of the equation defining the adjustment coefficient:

$$\lambda E\,(e^{RX}) = \lambda E(1 + RX + (RX)^2/2! + \dots)$$

Since the individual claim sizes, X, take positive values, the terms on the RHS are all positive.

So, ignoring terms in powers higher than X^2 gives:

$$\lambda + cR > \lambda[1 + RE(X) + \frac{R^2}{2}E(X^2)]$$

Subtracting λ from both sides:

$$cR > \lambda[RE(X) + \frac{R^2}{2}E(X^2)]$$

Dividing by R (which must be a positive number):

$$c > \lambda[E(X) + \frac{R}{2}E(X^2)]$$

Finally, we can now rearrange to get an inequality for R:

$$R < \frac{2[\frac{c}{\lambda} - E(X)]}{E(X^2)}$$

Q3. **Ruin Theory**

Claims on a portfolio of insurance policies arrive as a Poisson process with parameter λ, claim amounts having a Normal distribution with mean μ and variance σ^2, and there is a loading θ on premiums. The insurance company has an initial surplus of U.

(i) Explain carefully the meaning of $\Psi(U)$, $\Psi(U, t)$ and $\Psi(U,1)$.
(ii) State four factors that affect the size of $\Psi(U, t)$, for a given t.
(iii) Explain, for each factor, what happens to $\Psi(U, t)$ when the factor increases.

Sarah, the insurance company's actuary, prefers to consider the probability of ruin in discrete rather than continuous time.

(iv) Explain an advantage and disadvantage of Sarah's approach.

Solution:

(i) $\Psi(U) = P(U(t) < 0)$, for some $t, 0 < t < \infty$
$\Psi(U,t) = P(U(\tau) < 0)$, for some $\tau, 0 < \tau \leq t$
$\Psi(U,1) = P(U(t) < 0)$, for some $t, 0 < t \leq 1$

(ii) λ, μ, σ^2, θ and initial surplus U

(iii) higher λ increases $\Psi(U, t)$ as the process is faster – claims and premiums come in quicker

higher μ increases $\Psi(U, t)$ as claims amounts are larger, relative to the surplus held

higher θ reduces $\Psi(U, t)$ as premiums increase at a quicker rate, so more of a buffer

higher U reduces $\Psi(U, t)$ as more of a buffer to withstand claims

higher σ^2 will typically increase $\Psi(U, t)$, assuming that expected premiums are higher than expected claims, since the likelihood of more extreme claims increase, [but may reduce $\Psi(U, t)$ if expected claims are higher than expected premiums.]

(iv) Advantage – easier to measure, more useful for reporting

Disadvantage – less information, artificial, can miss time when ruin occurs.

***Q4. Excel-based**

An insurance company has a portfolio of policies. Consider an aggregate claims process, $S(t)$, where:

- $S(t) = X_1 + X_2 + ... + X_{N(t)}$
- $N(t)$ = the number of claims generated by the portfolio in the time interval $[0, t]$
- X_i = the amount of the i^{th} claim.

You have been given a single realisation of $N(5)$ and the values of the first three claim amounts, which follow an exponential distribution with mean 1.

(i) Calculate the value of the aggregate claims process at time $t = 5$.

Assume that c, the rate of premium income per unit time, is 1.1 and U, the initial surplus, is 0.5.

(ii) Calculate the value of the surplus process, $U(t)$, at time $t = 5$.

$N(t)$ is a Poisson process with $\lambda = 1$, therefore the interval of time between claims is exponentially distributed with a mean of 1. You have been given realisations of the time intervals t_i between the first three claims, where t_1 is the time of the first claim and t_i (for $i > 1$) is the time between claim i and claim $i - 1$.

(iii) Plot a chart showing the surplus process from time $t = 0$ to time $t = 5$.

(iv) Calculate the minimum value of U, to three decimal places, that would avoid ruin before time $t = 5$ in the process in part (iii).

See solutions in Excel spreadsheet on QMplus.