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MTH5126 Statistics for Insurance

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Week 9

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Ruin theory (continued)

The effect of changing parameter values

A formula for $\psi(U)$ when X is exponential

$\psi(U, t)$ as a function of t

Ruin probability as a function of initial surplus

Ruin probability as a function of loading factor

Ruin probability as a function of the Poisson parameter

Concluding remarks

Example Question

Valuing basic guarantees using simulation

- Example question

The effect of changing parameter values

We will now discuss the effect of changing parameter values on $\psi(U, t)$ and $\psi(U)$.

- We use the same basic assumptions as before. In particular we assume the aggregate claims process is a compound Poisson process. We also assume:

1. the Poisson parameter for the number of claims is 1
2. the expected value of an individual claim is 1
3. individual claims have an exponential distribution

- The impact of the first assumption is that the unit of time has been chosen to be such that the expected number of claims in a unit of time is 1.
- Hence $\psi(U, 500)$ is the probability of ruin (given initial surplus U) over a time period in which 500 claims are expected.
- The actual number of claims over this time period has a Poisson distribution (with parameter 500) and could take any non-negative integer value.

The effect of changing parameter values

- The implication of the second assumption is that the monetary unit has been chosen to be equal to the expected amount of a single claim.
- Hence $\psi(20, 500)$ is the probability of ruin (over a time period in which 500 claims are expected) given an initial surplus equal to 20 times the expected amount of a single claim.
- The advantage of using an exponential distribution for individual claims (Assumption 3) is that both e^{-RU} and $\psi(U)$ can be calculated for these examples.

A formula for $\psi(U)$ when X is exponential

- The formula for $\psi(U)$ when individual claim amounts are exponentially distributed with mean 1 and when the premium loading factor is θ is given by the following result:

When $F(x) = 1 - e^{-x}$:

$$\psi(U) = \frac{1}{1+\theta} e^{-\frac{\theta U}{1+\theta}}$$

✓ The IFoA syllabus does not require you to derive this result.

$\psi(U, t)$ as a function of t

There are some important features to note:

- $\psi(U,t)$ is an increasing function of t
- for small values of t, $\psi(U,t)$ increases very quickly
- for larger values of t, $\psi(U,t)$ increases less quickly and approaches asymptotically the value of $\psi(U)$

Ruin probability as a function of initial surplus

This time the features to note are:

1. Increasing the value of U decreases the value of $\psi(U, t)$ for any value of t.
2. $\psi(U)$ is a non-increasing function of U. In the case of exponential claim amounts, $\psi(U)$ is a decreasing function of U.

- Here we note that in the case of exponentially distributed individual claim amounts, the derivative with respect to U of $\psi(U)$ is:
$$\frac{d}{dU}\psi(U) = \frac{-\theta}{1+\theta}\psi(U)$$
This is negative since $\theta > 0$. Hence $\psi(U)$ is a decreasing function of U.
- It should be intuitively clear that $\psi(U, t)$ (of which $\psi(U)$ is a special case) should be a decreasing function of U.
- An increase in U represents an increase in the insurer's surplus without any corresponding increase in claim amounts.
- Thus, an increase in U represents an increase in the insurer's security and hence will reduce the probability of ruin.

Ruin probability as a function of premium loading

The main features to note are:

1. Increasing the value of θ decreases the value of $\psi(U,t)$ for any given value of t. This is an obvious result since an increase in θ is equivalent to an increase in the rate of premium income with no change in the aggregate claims process.
2. By general reasoning $\psi(U)$ must be a non-increasing function of θ. In the case of exponential claim amounts, $\psi(U)$ is a decreasing function of θ.

- For exponential claim amounts we have:

$$\frac{d}{d\theta}\psi(U) = -\frac{1}{1+\theta}\psi(U) - \frac{U}{(1+\theta)^2}\psi(U)$$

- This is clearly negative since θ, U and $\psi(U)$ are all positive quantities.
- Since the derivative is less than zero for all values of θ, $\psi(U)$ is a decreasing function of θ.

Ruin probability as a function of the Poisson parameter

Consider two risks:

Risk 1: aggregate claims are a compound Poisson process with Poisson parameter 1 and $F(x) = 1 - e^{-x}$.

The premium income per unit time to cover this risk is $(1 + \theta)$.

Risk 2: aggregate claims are a compound Poisson process with Poisson parameter 0.5 and $F(x) = 1 - e^{-x}$.

The premium income per unit time to cover this risk is $0.5(1 + \theta)$.

- The unit of time is taken to be one year. The only difference between these two risks is that twice as many claims are expected each year under Risk 1.
- Consider Risk 2 over a new time unit equivalent to two years.
- Then the distribution of aggregate claims and the premium income per unit time are now identical to the corresponding quantities for Risk 1.
- Hence, the probability of ruin over an infinite time span is the same for both risks.

Ruin probability as a function of the Poisson parameter

For illustration of the next point, let $\theta = 0.2$ and $U = 15$. Consider the aggregate claims process specified on slide 3. Using numerical method (beyond the IFoA syllabus), we can see that $\psi(15, t)$ is more or less constant for values of $t > 150$, i.e.

$$\psi(15, 150) \approx \psi(15)$$

- Consider a second aggregate claims process, which is the same as the process considered already except that its Poisson parameter is 150 and not 1.
- We use ψ^* to denote ruin probabilities for the second process and ψ, as before, to denote ruin probabilities for the original process. Note that these two processes are essentially identical with just the time unit changing. This change in time unit means that for any $t \geq 0$:

  $$\psi^*(U, t) = \psi(U, 150t)$$

  but it has no effect on the probability of ultimate ruin (put $t = \infty$ in the relationship above.)
- From this and the previous relations we can see that:

$$\psi^*(15, 1) = \psi(15, 150) \approx \psi(15) = \psi^*(15)$$

Ruin probability as a function of the Poisson parameter

- In words this is saying that for the second process, starting with initial surplus of 15, the probability of ruin within one time period is almost the same as the probability of ultimate ruin.
- This conclusion depends, crucially, on the fact that $\psi^*(15, 1)$ is a continuous time probability of ruin. To see this, consider $\psi_1^*(15, 1)$, which is just the probability that for the second process the surplus at the end of one time period is negative.
- $\psi_1^*(15, 1)$ can be calculated approximately by assuming that the aggregate claims in one time period, which we denote by $S^*(1)$, have a normal distribution.
- Recall then that individual claims have an exponential distribution with mean 1 and the number of claims in one time period has a Poisson distribution with mean 150.
- From this:

$$E[S^*(1)] = 150 \text{ and } var[S^*(1)] = 300$$

(or $\lambda m_1 = 150 \times 1, \lambda m_2 = 150 \times 2$ since $E(X^2) = 2$ for an $Exp(1)$ distribution).

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Ruin probability as a function of the Poisson parameter

$S^*(1) \sim Normal(150, 17.32^2)$, $\quad \dfrac{S^*(1) - 150}{17.32} \sim Normal(0, 1)$

Surplus at the end of one time period, $U(1) = U$ + premium income – $S^*(1)$

$= 15 + 1.2 \times 150 - S^*(1)$,

where premium income = $(1 + \theta)\, \lambda m_1$.

Finally:

$$\psi^*(15, 1) = P\,(U(1) < 0) = P\,(15 + 1.2 \times 150 - S^*(1) < 0)$$

$$= P(S^*(1) > 15 + 1.2 \times 150)$$

$$= P(\,[S^*(1) - 150] \,/\, 17.32 > [195 - 150] \,/\, 17.32)$$

$= P(Z > 2.598)$, where Z is the standard normal r.v.

$$\approx 0.005$$

Concluding remarks

- When individual claim amounts are exponentially distributed with mean 1, first note that if $\theta = 0$ then $\psi(U) = 1$ irrespective of the value of U.
- This result is in fact true for any form of $F(x)$.
- It trivially follows that if $\theta < 0$, $\psi(U) = 1$.
- In other words, a positive premium loading is essential if ultimate ruin is not to be certain.
- Also note that throughout this section it has been assumed that individual claim amounts are exponentially distributed with mean 1.
- This mean could be measured in units of £100, £1,000 or even £1,000,000.
- The parameter of the exponential distribution can still be set to 1 without loss of generality, provided that the monetary unit is correctly specified.

Concluding remarks

- In simple terms, the probability of ruin when U is £1 is the same as the probability of ruin when U is 100 pence.
- It can be said that:

$$\psi(U) \text{ when } F(x) = 1 - e^{-\alpha x}$$

is the same as:

$$\psi(\alpha U) \text{ when } F(x) = 1 - e^{-x}$$

- In other words, if the expected claims per unit time increase by a factor of α so too must the initial surplus if the probability of ultimate ruin is to be unchanged.

Example Question

A general insurance company is planning to set up a new class of travel insurance. It plans to start the business with £2 million and expects claims to occur according to a Poisson process with parameter 50.

Individual claims are thought to have a gamma distribution with parameters $\alpha = 150$ and $\beta = 1/4$.

A premium loading factor of 30% is applied.

Explain how each of the following changes to the company's model will affect the probability of *ultimate ruin*:

i. A 28% premium loading factor is applied instead

ii. Individual claims are found to have a gamma distribution with parameters $\alpha = 150$ and $\beta = 1/2$.

iii. The Poisson parameter is now believed to be 60.

Answer:

i) Since there is a smaller loading factor, the premiums will be reduced even though claims remain the same. Hence, the probability of ultimate ruin will increase.

Example Question

Answer (continued):

(ii)

The mean of the distribution has decreased from 600 to 300 and the variance has decreased from 2,400 to 600.

Therefore, the claims are smaller on average and less uncertain. Both of these factors will decrease the probability of ultimate ruin.

(iii)

- The Poisson parameter has increased so claims occur more often (but their size is unchanged).
- However, the premium received will also increase proportionally, since $c = (1 + \theta)\lambda m_1$.
- Hence the timing at which ruin may occur will be earlier, but not the probability of it occurring in the first place.

Therefore, the probability of ultimate ruin will remain unchanged.