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MTH5126 Statistics for Insurance

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Week 8

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Ruin theory

Introduction

- Up until now we have studied aggregate claims over a single time period.
- We will now take this a step further by considering claims generated by a portfolio over **successive time periods**.
- We need some new notation:
  - $N(t)$ is the number of claims generated by the portfolio in the time interval $[0, t]$, for all $t \geq 0$
  - X_i is the amount of the i^{th} claim, $i = 1, 2, 3, ...$
  - $S(t)$ is the aggregate claims in the time interval $[0, t]$ for all $t \geq 0$

$\{X_i\}_{i=1}^{\infty}$ is a sequence of random variables.

$\{N(t)\}_{t\geq 0}$ is a family of random variables, one for each time $t \geq 0$.

$\{S(t)\}_{t\geq 0}$ is a family of random variables, one for each time $t \geq 0$.

So both $\{N(t)\}_{t\geq 0}$ and $\{S(t)\}_{t\geq 0}$ are stochastic processes.

Ruin theory

Introduction

It can be seen that:

$$S(t) = \sum_{i=1}^{N(t)} X_i$$

With the understanding that $S(t)$ is zero if $N(t)$ is zero.

- The stochastic process $\{S(t)\}_{t \geq 0}$ is known as the aggregate claims process for the risk.
- The random variables $N(1)$ and $S(1)$ represent the number of claims and the aggregate claims respectively from the portfolio in the first unit of time.
- These two random variables correspond to what we have been calling N and S, respectively.
- So we have just taken the idea of a compound distribution and generalised it to cover different time periods.

Ruin theory

Introduction

- The insurer of the portfolio will receive premiums from the policyholders.
- We will assume throughout this section that premium income is received continuously and at a constant rate:
  - ***c* = the rate of premium income per unit time**

- This means that the total income we receive in the time interval $[0, t]$ is ct.
- We will also assume that c is strictly positive.

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The surplus process

- Suppose that at time 0 the insurer has an amount of money set aside for a particular portfolio.
- This amount of money is called the initial surplus and is denoted by U. It will always be assumed that $U \geq 0$.
- The insurer needs this initial surplus because the future premium income on its own may not be sufficient to cover the future claims. Here we are ignoring expenses.
- The insurer's surplus at any given future time t with $t > 0$ is a random variable since its value depends on the claims experience up to time t.
- The insurer's surplus at time t is denoted by $U(t)$. The following formula for $U(t)$ can be written:

$$U(t) = U + ct - S(t)$$

- In words this says that the insurer's surplus at time t is the initial surplus, plus the premium income up to time t, minus the aggregate claims up to time t.

The surplus process

- Notice that **the initial surplus and the premium income are not random variables** since they are determined before the risk process starts.
- The surplus formula is valid for $t \geq 0$ with the understanding that $U(0) = U$.
- For a given value of t, $U(t)$ is a random variable because $S(t)$ is a random variable.
- Hence $\{U(t)\}_{t \geq 0}$ is a stochastic process, which is known as the cash flow process or surplus process.

The probability of ruin in continuous time

When surplus falls below zero the insurer has run out money we say that ***ruin*** has occurred.

- In our simplified model, the insurer will want to keep the probability of this event, that is, the probability of ruin, as small as possible.
- Ruin can be thought of as meaning insolvency in this instance (in practice determining whether or not an insurance company is, in fact, insolvent, is quite complex).
- Another way of looking at the probability of ruin is to think of it as the probability that, at some future time, the insurance company will need to provide more capital to finance this particular portfolio.

The probability of ruin in continuous time

Being more precise we define two probabilities:

$$\psi(U) = P[U(t) < 0, \text{ for some t}, 0 < t < \infty]$$
$$\psi(U,\ t) = P[U(\tau) < 0, \text{ for some } \tau,\ 0 < \tau < t]$$

- $\psi(U)$ is the probability of ultimate ruin (given initial surplus U) and
- $\psi(U, t)$ is the probability of ruin within time t (given initial surplus U).

These probabilities are sometimes referred to as **the probability of ruin in infinite time** and **the probability of ruin in finite time**.

The probability of ruin in continuous time

- Here are some important logical relationships between these two probabilities for $0 < t_1 \leq t_2 < \infty$ and for $0 \leq U_1 \leq U_2$.

  1. $\psi(U_2, t) \leq \psi(U_1, t)$
  2. $\psi(U_2) \leq \psi(U_1)$
  3. $\psi(U, t_1) \leq \psi(U, t_2) \leq \psi(U)$
  4. $\lim_{t \longrightarrow \infty} \psi(U, t) = \psi(U)$

Question. What is the $\lim_{U \to \infty} \psi(U, t)$?

Answer: Zero! With unlimited initial surplus we won't ever have to worry about ruin!

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The probability of ruin in discrete time

- The two probabilities of ruin we have considered so far have been in continuous time but in practice it may only be possible (or even desirable) to check for ruin at discrete time intervals.

For a given interval of time, denoted by *h*, the following two discrete time probabilities of ruin are defined:

$$\psi_h(U) = P[U(t) < 0, \text{ for some t }, t = h, 2h, 3h, \ldots]$$

$$\psi_h(U, t) = P[U(\tau) < 0, \text{ for some } \tau, \tau = h, 2h, 3h, \ldots, t-h, t]$$

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The probability of ruin in discrete time

- Here are some important logical relationships between these two probabilities for $0 \leq t_1 \leq t_2 < \infty$ and for $0 \leq U_1 \leq U_2$.

1. $\psi_h(U_2, t) \leq \psi_h(U_1, t)$
2. $\psi_h(U_2) \leq \psi_h(U_1)$
3. $\psi_h(U, t_1) \leq \psi_h(U, t_2) \leq \psi_h(U)$
4. $\lim_{t \to \infty} \psi_h(U, t) = \psi_h(U)$
5. $\psi_h(U, t) \leq \psi(U, t)$

Regarding the relationship 5:

$\psi(U, t)$ involves checking for ruin at all possible times(up to t). Since the more often we check for ruin, the more likely we are to find it, we would expect that $\psi(U, t)$would be greater than (or equal to) $\psi_h(U, t)$.

The Poisson process

In this section we will make some assumptions about the claim number process $\{N(t)\}_{t\geq 0}$ and the claim amounts process $\{S(t)\}_{t\geq 0}$ and the claim amounts $\{X_i\}_{i=1}^{\infty}$

- The claim number process will be assumed to be a Poisson process, leading to a compound Poisson process $\{S(t)\}_{t\geq 0}$ for aggregate claims.
- The assumptions made here will hold for the rest of the module.
- The Poisson process is an example of a counting process. Here the number of claims arising from a risk is the quantity of interest.
- Since the number of claims is being counted over time, the claim number process $\{N(t)\}_{t\geq 0}$ must satisfy the following conditions:
  1. $N(0) = 0$, *i.e.,* there are no claims at time 0
  2. for any $t > 0$, $N(t)$ must be integer valued
  3. when $s < t$, $N(s) \leq N(t)$, *i.e.,* the number of claims over time is non-decreasing
  4. when $s < t$, $N(t) - N(s)$ represents the number of claims occurring in the time interval $(s, t]$

The Poisson process

- The claim number process $\{N(t)\}_{t\geq 0}$ is defined to be a Poisson process with parameter λ if the following conditions are satisfied:
  1. $N(0) = 0$ and $N(s) \leq N(t)$ when $s < t$
  2. $P(N(t+h) = r \mid N(t) = r) = 1 - \lambda h + o(h)$

     $P(N(t+h) = r+1 \mid N(t) = r) = \lambda h + o(h)$

     $P(N(t+h) > r+1 \mid N(t) = r) = o(h)$
  3. When $s < t$, the number of claims in the time interval $(s, t]$ is independent of the number of claims up to time s

- Note that a function $f(x)$ is described as being $o(x)$ as x goes to zero, if:

$$\lim_{x \to 0} \frac{f(x)}{x} = 0$$

The Poisson process

Time to the first claim

- When studying a Poisson process the distribution of the time to the first claim and the times between claims is often of particular interest.

Let the random variable, T_1 denote the time of the first claim. Then, for a fixed value of t, if no claims have occurred by time t, $T_1 > t$.

Hence:

$$P(T_1 > t) = P(N(t) = 0) = e^{-\lambda t},$$

since $N(t)$ has a Poisson distribution with parameter λt. And

$$P(T_1 \leq t) = 1 - e^{-\lambda t}$$

So T_1 has an exponential distribution with parameter λ.

The Poisson process

Time between claims

For $i = 1, 2, 3\ldots$ let the random variable T_i denote the time between the $(i-1)^{th}$ and the i^{th} claims. Then:

$$P(T_{n=1} > t + r | \textstyle\sum_{i=1}^{n} T_i = r) = \mathrm{P}(\displaystyle\sum_{i=1}^{n+1} T_i = t + r \,|\, \sum_{i=1}^{n} T_i = r)$$

$$= P(N(t + r) = n | N(r) = n)$$
$$= P(N(t + r) - N(r) = 0 | N(r) = n).$$

Using the fact that claim numbers in different time periods are independent we get:

$$P(N(t + r) - N(r) = 0 | N(r) = n) = P(N(t + r) - N(r) = 0)$$

Finally, since the number of claims in a time interval of length t does not depend on when that time interval starts:

$$P[N(t + r) - N(r) = 0] = P(N(t) = 0) = e^{-\lambda t}$$

So the time between claims (known as inter-event times) also have an exponential distribution with parameter λ.

The Poisson process

Example

If reported claims follow a Poisson process with rate 5 per day (and the insurer has a 24 hour hotline), calculate:

i. The probability that there will be fewer than 2 claims reported on a given day

ii. The probability that another claim will be reported during the next hour.

Answer:

i) The expected number of claims reported on a given day is 5 so the number of claims reported on a given day has a *Poisson*(5) distribution and the probability that there will be fewer than 2 claims is:

$$P(N < 2)$$
$$= P(N = 0) + P(N = 1)$$
$$= e^{-5} + 5e^{-5}$$
$$= 0.040 \ (or\ 4\%)$$

The Poisson process

Example

If reported claims follow a Poisson process with rate 5 per day (and the insurer has a 24 hour hotline), calculate:

(i)The probability that there will be fewer than 2 claims reported on a given day

(ii)The probability that another claim will be reported during the next hour.

Answer (continued):

(ii) The waiting time until the next event has an *Exponential*(5) distribution. So the probability that there will be a claim during the next hour (= $\frac{1}{24}$) of a day is:

$$P(T < t) = 1 - e^{-\lambda t}$$

$$P(T < \tfrac{1}{24}) = 1 - e^{-5(1/24)} = 0.1881 \ (or\ 18.8\%)$$

The compound Poisson process

We now combine the Poisson process for the number of claims with a claim amount distribution to give a compound Poisson process for the aggregate claims.

- We will make the following three, important, assumptions:
  1) the random variables $\{X_i\}_{i=1}^{\infty}$ are independent and identically distributed
  2) the random variables $\{X_i\}_{i=1}^{\infty}$ are independent of $N(t)$ for all $t \geq 0$
  3) the stochastic process $\{N(t)\}_{t\geq 0}$ is a Poisson process whose parameter is denoted λ

The last assumption means that for any $t \geq 0$, the random variable $N(t)$ has a Poisson distribution with parameter λt, so that:

$$P(N(t) = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, \quad k = 0, 1, 2, \ldots$$

With these assumptions the aggregate claims process $\{S(t)\}_{t\geq 0}$ is called a compound Poisson process with Poisson parameter λ.

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The compound Poisson process

So then for a fixed value of t, $S(t)$ has a compound Poisson distribution with Poisson parameter λt.

- Note the terminology change: "Poisson parameter λ" becomes "Poisson parameter λt" when we switch from the process to the distribution.

- The common distribution function of the X_i's will be denoted by $F(x)$ and it will be assumed that $F(0) = 0$ so that all claims are for positive amounts.
- The probability density function of the X_i's, if it exists, will be denoted by $f(x)$ and the k^{th} moment about zero of the X_i's, if it exists, will be denoted by m_k, so that:

$$m_k = E[X^k], k = 1, 2, 3, \ldots$$

- Whenever the common moment generating function of the X_i's exists, its value at the point r will be denoted by $M_X(r)$.

The compound Poisson process

Since, for a fixed value of t, $S(t)$ has a compound Poisson distribution it follows that the process $\{S(t)\}_{t\geq 0}$ has mean $\lambda t m_1$, variance $\lambda t m_2$ and moment generating function $M_S(r)$, where:

$$M_S(r) = exp\{\lambda t(M_X(r) - 1\}$$

- For the rest of our ruin theory conversations, we now make the following (intuitively reasonable) assumption about the rate of premium income.

$$c > \lambda m_1$$

Question: Why is this intuitively reasonable?

Answer: So that the insurer's premium income (per unit time) is greater than the expected claims outgo (per unit time).

The compound Poisson process

Example

The aggregate claims arising during each year from a particular type of annual insurance policy are assumed to follow a normal distribution with mean $0.7P$ and standard deviation $2.0P$, where P is the annual premium.

Claims are assumed to arise independently, and insurers assess their solvency position at the end of each year.

A small insurer with an initial surplus of £0.1m expects to sell 100 policies at the beginning of the coming year in respect of identical risks for an annual premium of £5,000. The insurer incurs expenses of $0.2P$ at the time of writing each policy.

Calculate the probability that the insurer will prove to be insolvent at the end of the coming year. (You can ignore interest.)

The compound Poisson process

Example

Answer: Using the information given in the question we can see that the insurer's surplus at the end of the coming year will be:

$$U(1) = \text{initial surplus} + \text{premiums} - \text{expenses} - \text{claims}$$
$$= 0.1m + 100 \times 5,000 - 100 \times 0.2 \times 5,000 - S(1)$$
$$= 0.5m - S(1)$$

The distribution of $S(1)$ is:

$$S(1) \sim N(100\times0.7\times5{,}000,\ 100\times(2.0\times5{,}000)^2) = N(0.35m,\ (0.1m)^2)$$

So the probability that the surplus will be negative is:

$$P(U(1) < 0) = P(S(1) > 0.5m) = P(N(0.35m, (0.1m)^2) > 0.5m)$$
$$= 1 - \Phi\left(\frac{0.5m - 0.35m}{0.1m}\right) = 1 - 0.93319 = 0.067$$

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The compound Poisson process

Mean, variance and MGF of the compound Poisson process

For a compound Poisson process $S(t)$, the mean and variance of the total claim amount are given by:

$$E(S(t)) = \lambda t\, E(X)$$
$$var(S(t)) = \lambda t\, E(X^2)$$

The moment generating function of the process is given by:

$$M_{S(t)}(r) = e^{\lambda t\,(M_X(r)-1)}$$

Premium security loadings

- So far, we have used c to denote the rate of premium income per unit time, independent of the claims outgo.
- In some cases, it is more useful to think of the rate of premium income being related to that of claims outgo.
- For the insurer to survive, the rate at which premium income comes in needs to be greater than the rate at which claims are paid out.

If this is not true, then the insurer is certain to be ruined at some point. So we will sometimes write c as:

$$c = (1 + \theta)\lambda m_1$$

where $\theta\,(> 0)$ is the premium loading factor.

Lundberg's inequality

Lundberg's inequality states that:

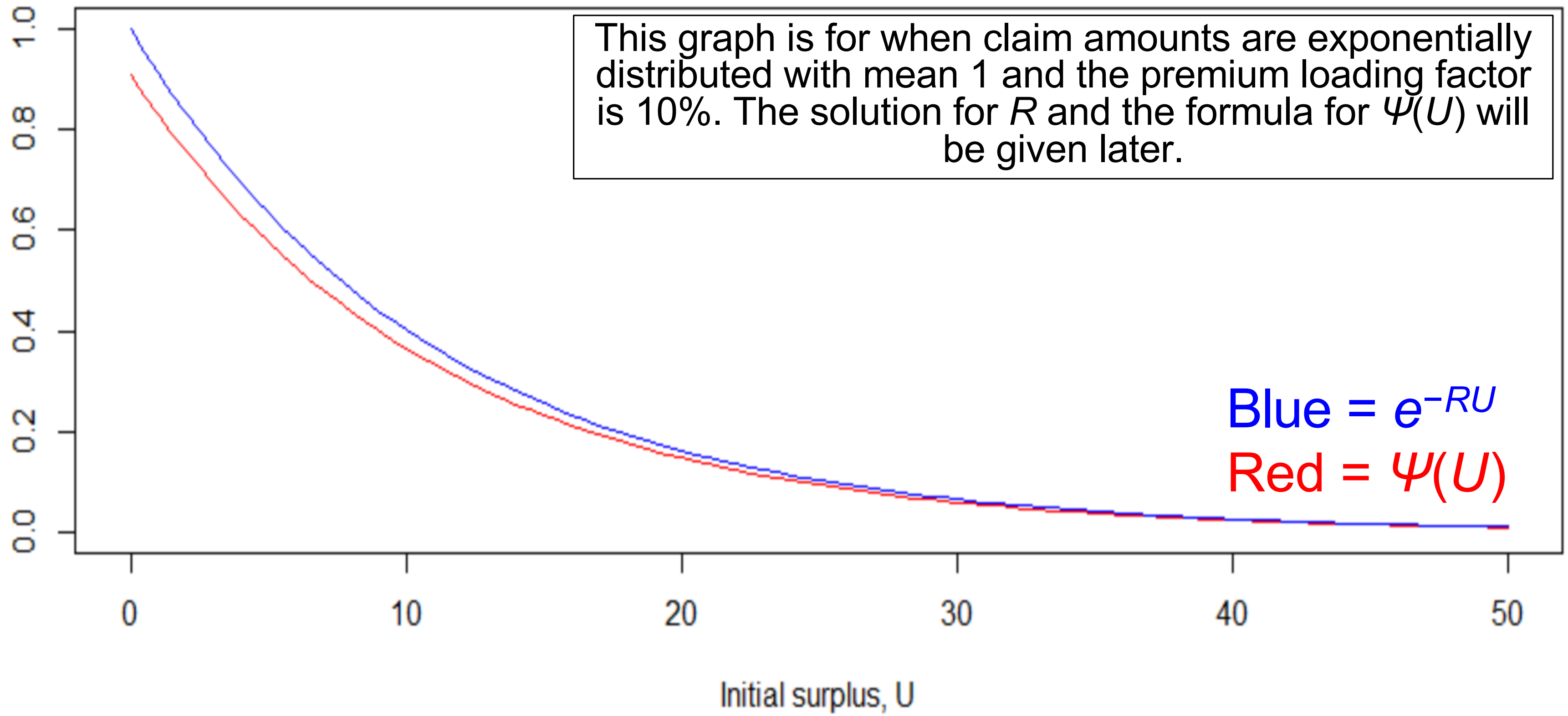
$$\psi(U) \leq e^{-RU}$$

where U is the insurer's initial surplus and $\psi(U)$ is the probability of ultimate ruin.

- R is a parameter associated with a surplus process known as the **adjustment coefficient**.
- Its value depends on the distribution of aggregate claims and on the rate of premium income.
- Before defining R, we will look at the importance of this result and some features of the adjustment coefficient.

Lundberg's inequality

Pictorial view



Lundberg's inequality

Interpretation

- We can use e^{-RU} as an approximation to $\psi(U)$.
- R can be interpreted as measuring risk (an inverse measure of risk).
- The larger the value of R, the smaller the upper bound for $\psi(U)$ will be. Hence $\psi(U)$ would be expected to decrease as R increases.
- R is a function of the parameters that affect the probability of ruin and R's behaviour as a function of these parameters can be observed.

Lundberg’s inequality

R as a function of the loading factor, θ

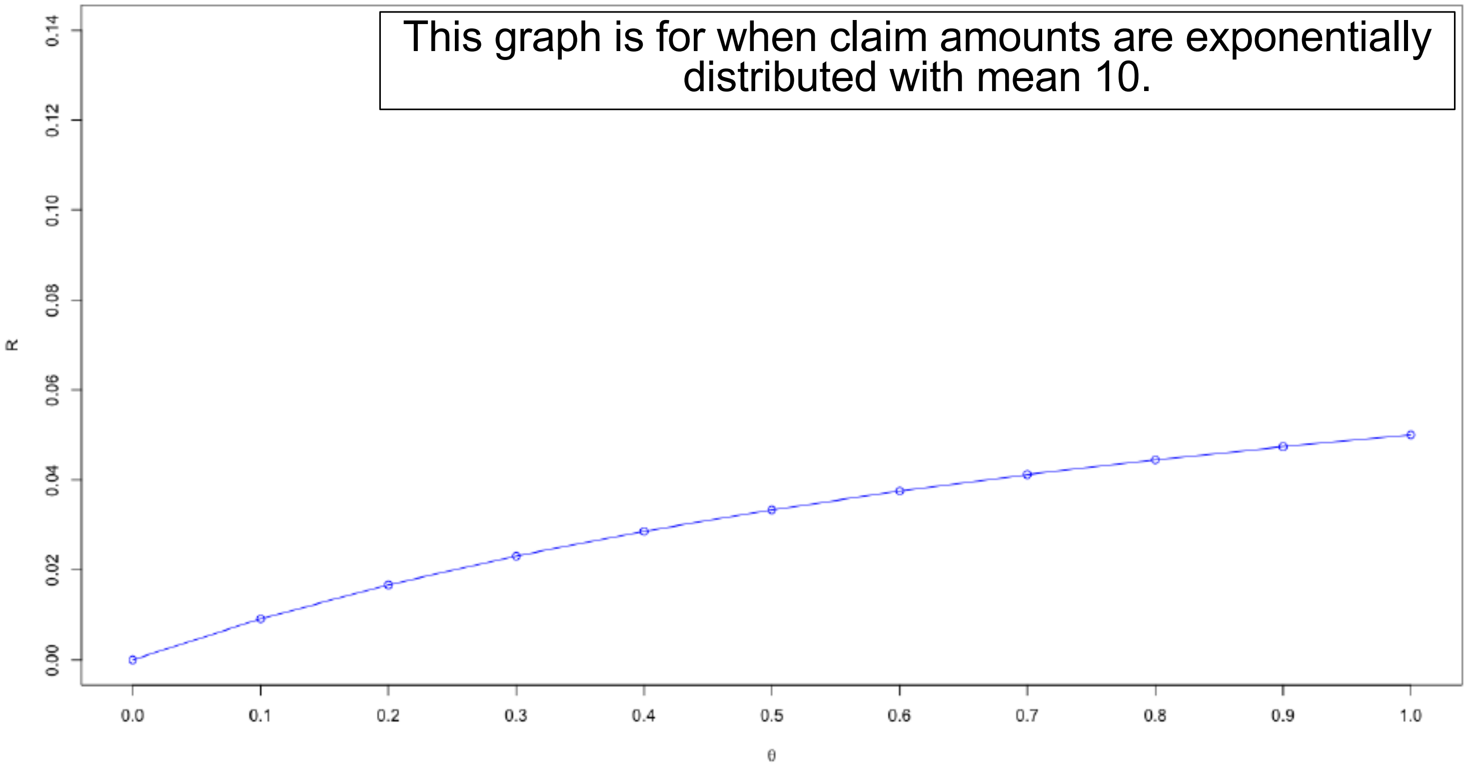
- We will now take a look at a graph of R as a function of the loading factor, θ, when the claim amount distribution is exponential with mean 10. [see graph on next slide]
- We can see that R is an increasing function of θ.
- This is not surprising since $\psi(U)$ would be expected to be a decreasing function of θ.
- Also, since:

$$\psi(U) \approx e^{-RU}$$

any factor causing a decrease in $\psi(U)$ would cause R to increase.

Lundberg's inequality

R as a function of the loading factor, θ



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The adjustment coefficient

- The surplus process depends on: the initial surplus, the aggregate claims process and on the rate of premium income.
- The adjustment coefficient, R, is a parameter associated with a surplus process which takes account of two of these factors: aggregate claims and premium income.
- R gives a measure of risk for a surplus process.
- When aggregate claims are a compound Poisson process, R is defined in terms of the Poisson parameter, the moment generating function of individual claim amounts and the premium income per unit time.

The adjustment coefficient

The adjustment coefficient, denoted R is defined to be the unique positive root of:

$$\lambda M_X(R) - \lambda - cR = 0$$

So, R is given by:

$$\lambda M_X(R) = \lambda + cR$$

- Note that this equation implies that the value of R depends on the Poisson parameter, the individual claim amount distribution and the rate of premium income.

However, writing $c = (1 + \theta)\lambda m_1$ gives:

$$M_X(R) = 1 + (1 + \theta)m_1 R$$

so that R is independent of the Poisson parameter and simply depends on the loading factor θ and the individual claim amount distribution.

The adjustment coefficient

Example 1

An insurer knows from past experience that the number of claims received per month has a Poisson distribution with mean 15 and that claim amounts have an exponential distribution with mean 500. The insurer uses a security loading of 30%.

Calculate the insurer's adjustment coefficient and give an upper bound for the insurer's probability of ruin, if the insurer sets aside an initial surplus of £1,000.

Answer:

The equation for the adjustment coefficient is: $M_X(R) = 1 + (1 + \theta)m_1R$

We have $X \sim Exp(\frac{1}{500})$ so that $M_X(R) = \frac{1}{1-500R}$

$\theta = 0.3$ and $m_1 = E(X) = 500$

We can substitute these values into the equation to get:

$$\frac{1}{1-500R} = 1 + 1.3 \times 500R = 1 + 650R$$

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The adjustment coefficient

Example 1

Answer (continued):

Rearranging this we get:

$$1 = (1 - 500R)(1 + 650R)$$
$$1 = 1 - 500R + 650R - 325,000R^2$$
$$0 = 150 - 325,000R$$
$$\Rightarrow \quad R = 0.000462$$

From Lundberg's inequality, $\psi(U) \leq e^{-RU}$, we get:

$$\psi(U) \leq e^{-0.000462 \times 1,000} = 0.630$$

It is worth noting that the Poisson parameter was not used in the solution.

The adjustment coefficient

When individual claims are exponentially distributed

Now consider the exponential distribution: We have $F(x) = 1 - e^{-\alpha x}$

(Note that here we are using α as the parameter for the exponential distribution to avoid confusion with the Poisson parameter, λ.)

For this distribution we have $M_X(R) = \frac{\alpha}{\alpha - R}$ so:

$$\lambda + cR = \frac{\lambda\alpha}{\alpha - R}$$

$$\implies \lambda\alpha - \lambda R + cR\alpha - cR^2 = \lambda\alpha$$

$$\implies R^2 - (\alpha - \frac{\lambda}{c})R = 0$$

$$\implies R = \alpha - \frac{\lambda}{c}$$

The adjustment coefficient

When individual claims are exponentially distributed

So now if we have:

$$c = \frac{(1+\theta)\lambda}{\alpha}$$

then:

$$R = \frac{\alpha\theta}{(1+\theta)}$$

since the mean of this distribution is $m_1 = \frac{1}{\alpha}$

The adjustment coefficient

Example 2

Write down the equation for the adjustment coefficient for personal accident claims if 90% of claims are for £10,000 and 10% of claims are for £25,000, assuming a proportional security loading of 20%.

Show that this equation has a solution in the range:

$$0.00002599 < R < 0.00002601$$

Answer:

The adjustment coefficient satisfies:

$$1 + (1 + \theta)m_1R = M_X(R)$$

The distribution of the individual claim sizes X is:

$$X = \begin{cases} 10,000 & \text{with probability} \quad 0.9 \\ 25,000 & \text{with probability} \quad 0.1 \end{cases}$$

The adjustment coefficient

Example 2

Answer (continued):

So:

$$E(X) = \sum xP(X = x) = 0.9 \times 10,000 + 0.1 \times 25,000 = 11,500$$

and:

$$M_X(R) = E(e^{RX)}) = \sum e^{Rx} P(X = x) = 0.9e^{10,000R} + 0.1e^{25,000R}$$

The security loading is $\theta = 0.2$.

The adjustment coefficient

Example 2

Answer (continued):
So, the adjustment coefficient equation is:

$$1 + 1.2 \times 11,500R = 0.9e^{10,000R} + 0.1e^{25,000R}$$

$$\Rightarrow \quad 1 + 13,800R = 0.9e^{10,000R} + 0.1e^{25,000R}$$

We can show that there is a solution in the range stated by looking at the values of LHS - RHS:

$$R = 0.00002599 \quad \Rightarrow$$

$$1 + 13,800R - (0.9e^{10,000R} + 0.1e^{25,000R}) = 0.000035$$

$$R = 0.00002601 \quad \Rightarrow$$

$$1 + 13,800R - (0.9e^{10,000R} + 0.1e^{25,000R}) = -0.000018$$

Since we have a reversal of signs and we are dealing with a continuous function, the difference must be zero at some point between these two values.

This means that there is a solution of the equation in the range $0.00002599 < R < 0.00002601$

The adjustment coefficient

An upper bound for R

If the equation for R has to be solved numerically, it is useful to have a rough idea of R's value. We can find a simple upper bound for R as follows:

$$\lambda + cR = \lambda M_X(R)$$

$$= \lambda \int_0^\infty e^{Rx} f(x)dx$$

$$> \lambda \int_0^\infty (1 + Rx + \frac{1}{2}R^2x^2) f(x)dx$$

(Because all terms in the series for e^{Rx} are positive. So e^{Rx} must always be greater than the total of the first few terms)

$$= \lambda(1 + Rm_1 + \frac{1}{2}R^2 m_2)$$

The adjustment coefficient

An upper bound for R

So that $(c - \lambda m_1)R > \frac{1}{2}\lambda R^2 m_2$, giving :

$$R < \frac{2(c - \lambda m_1)}{\lambda m_2}$$

So that $R < \frac{2\theta m_1}{m_2}$ when $c = (1 + \theta)\lambda m_1$

Notice that if the value of R is small, then it should be very close to this upper bound since the approximation to e^{Rx} should be good.

The adjustment coefficient

A lower bound for R

- A lower bound for R can be derived when there is an upper limit, say M, to the amount of an individual claim. E.g.: If individual claim amounts are uniformly distributed on (0,100), then M = 100.
- The result is proved in a similar way. The lower bound is found by applying the inequality:

$$e^{Rx} \leq \frac{x}{M}e^{RM} + 1 - \frac{x}{M}, \quad 0 \leq x \leq M$$

which is proved through the series expansion for e^{RM}:

$$\begin{aligned} \frac{x}{M}e^{RM} + 1 - \frac{x}{M} &= \frac{x}{M}\sum_{j=0}^{\infty}\frac{(RM)^j}{j!} + 1 - \frac{x}{M} \\ &= 1 + \sum_{j=1}^{\infty}\frac{R^j M^{j-1} x}{j!} \\ &\geq 1 + \sum_{j=1}^{\infty}\frac{(Rx)^j}{j!}, \quad 0 \leq x \leq M \\ &= e^{Rx} \end{aligned}$$

The adjustment coefficient

A lower bound for R

We can now show:

$$R > \frac{1}{M} ln \left(\frac{c}{\lambda m_1} \right)$$

This is true when individual claim amounts have a continuous distribution on $(0, M)$.

The starting point is the equation defining R:

$$\lambda + cR = \lambda \int_0^M e^{Rx} f(x) dx$$

$$\leq \lambda \int_0^M \left(\frac{x}{M} e^{RM} + 1 - \frac{x}{M} \right) f(x) dx$$

$$= \frac{\lambda}{M} e^{RM} m_1 + \lambda - \frac{\lambda}{M} m_1$$

The adjustment coefficient

A lower bound for R

Rearranging we get:

$$\frac{c}{\lambda m_1} \leq \frac{1}{RM}(e^{RM} - 1) = 1 + \frac{RM}{2} + \frac{(RM)^2}{3!} + \ldots < 1 + \frac{RM}{1!} + \frac{(RM)^2}{2!} + \cdots = e^{RM}$$

Taking logs and rearranging:

$$R > \frac{1}{M} ln\left(\frac{c}{\lambda m_1}\right)$$

as required.

We can find other approximation for R, especially when R is small, by truncating the series expansion of e^{Rx}.

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The adjustment coefficient

Summary of upper and lower bounds for R

Upper and lower bounds for R

Upper bound

$R < \frac{2(c-\lambda m_1)}{\lambda m_2}$ or $R < \frac{2\theta m_1}{m_2}$ (when $c = (1+\theta)\lambda m_1$)

Lower bound $(0 \leq X \leq M)$

$R > \frac{1}{M} ln\left(\frac{c}{\lambda m_1}\right)$ or $R > \frac{1}{M} ln(1+\theta)$ (when $c = (1+\theta)\lambda m_1$)