MTH5126 - Statistics for Insurance

Academic Year: 2022-23 **Semester: B**

Worksheet 4 - Solutions

Q1.

Claims on a group of policies of a certain type arise as a Poisson process with parameter λ_1.

Claims on a second, independent, group of policies arise as a Poisson process with parameter λ_2.

The aggregate claim amounts on the respective groups are denoted S_1 and S_2. Using moment generating functions, show that S (the sum of S_1 and S_2) has a compound Poisson distribution and hence derive the Poisson parameter for S.

Answer:

Let N_i denote the number of claims on policies of type i for $i = 1, 2$.

N_1 follows a Poisson distribution with Poisson parameter λ_1 and aggregate claims S_1 follows a compound Poisson distribution with Poisson parameter λ_1.

Similarly, N_2 follows a Poisson distribution with Poisson parameter λ_2 and aggregate claims S_2 follows a compound Poisson distribution with Poisson parameter λ_2.

Let X_i denote the claim amount random variables for policies of type i for $i = 1, 2$. Then:

$M_{S_i}(t) = M_{N_i}[lnM_{X_i}(t)]$, using the results for MGF of a compound distribution

$= exp\ (\lambda_i[M_{X_i}(t) - 1])$, using $M_{N_i}(t) = exp(\lambda_i[e^t - 1])$

By independence:

$M_S(t) = E(e^{tS}) = E(e^{t(S_1+S_2)}) = E\ (e^{tS_1}e^{tS_2}) = E(e^{tS_1})E(e^{tS_2}) = M_{S_1}(t)M_{S_2}(t)$

Hence:

$$M_S(t) = exp\ (\lambda_1[M_{X_1}(t) - 1])\ exp\ (\lambda_2[M_{X_2}(t) - 1])$$
$$= exp\ (\lambda_1 M_{X_1}(t) + \lambda_2 M_{X_2}(t) - (\lambda_1 + \lambda_2))$$
$$= exp\ (\lambda\,(M_X(t) - 1))$$

where:

$$\lambda = \lambda_1 + \lambda_2$$

and:

$$M_{\mathbf{X}}(t) = \frac{\lambda_1 M_{X_1}(t) \lambda_2 M_{X_2}(t)}{\lambda_1 + \lambda_2}$$

Q2.

Consider aggregate claims over a period of 1 year, S, on a portfolio of general insurance policies:

$$S = X_1 + X_2 + \cdots + X_N$$

The number of claims each year, N, has a Poisson distribution with mean 12. $X_1, X_2, \ldots$ are assumed to be random variables, independent of each other and independent of N, with the following distribution:

$$f(x) = 0.01e^{-0.01x}, \quad 0 < x < £200$$

$$P(X = £200) = e^{-2}$$

The insurer effects excess of loss reinsurance with a retention limit of £100. Annual aggregate claims paid by the reinsurer are denoted by S_R. Calculate $E(S_R)$.

Answer:

$S_R = Z_1 + Z_2 + \ldots + Z_N$ where Z_i is the amount paid on the ith claim by the reinsurer.

Using the formula for the mean of a compound Poisson distribution we get:

$E(S_R) = \lambda E(Z)$

$$Z = \begin{cases} 0 & X < 100 \\ X - 100 & 100 \le X < 200 \\ 100 & X = 200 \end{cases}$$

$$E(Z) = \int_{100}^{200} (x-100)0.01e^{-0.01x}dx + 100e^{-2}$$

We can integrate by parts to find the value of the integral:

$$\int_{100}^{200} (x-100)0.01e^{-0.01x}dx = \left[-(x-100)e^{-0.01x}\right]_{100}^{200} + \int_{100}^{200} e^{-0.01x}dx$$
$$= \left[-(x-100)e^{-0.01x} - 100e^{-0.01x}\right]_{100}^{200}$$
$$= 100e^{-1} - 200e^{-2}$$
$$\implies E(Z) = 100e^{-1} - 200e^{-2} + 100e^{-2} = 100(e^{-1} - e^{-2}) = 23.254416$$

Therefore:
$E(S_R) = \lambda E(Z) = 12 \times 23.2544 = 279.053$

Q3. R

In an insurance company's portfolio, individual claim sizes, in £, follow an exponential distribution with parameter 0.0001.

(i) Run the following code: `set.seed(123)` and then use R to simulate 10,000 claims and plot a histogram of the simulated data. Paste your R code and chart into your answer.

(ii) Calculate the mean and variance of the simulated claims in part (i).

The insurer decides to take out an individual excess of loss reinsurance arrangement with a retention level of £20,000.

(iii) Calculate the mean and the variance of the claims paid, under this arrangement, by

(a) the insurer
(b) the reinsurer.

The insurer wishes to determine an appropriate retention limit and has asked an analyst to investigate the effect of different retention limits on the mean and variance of claims.

(iv) Calculate the mean and variance of the claims paid by:

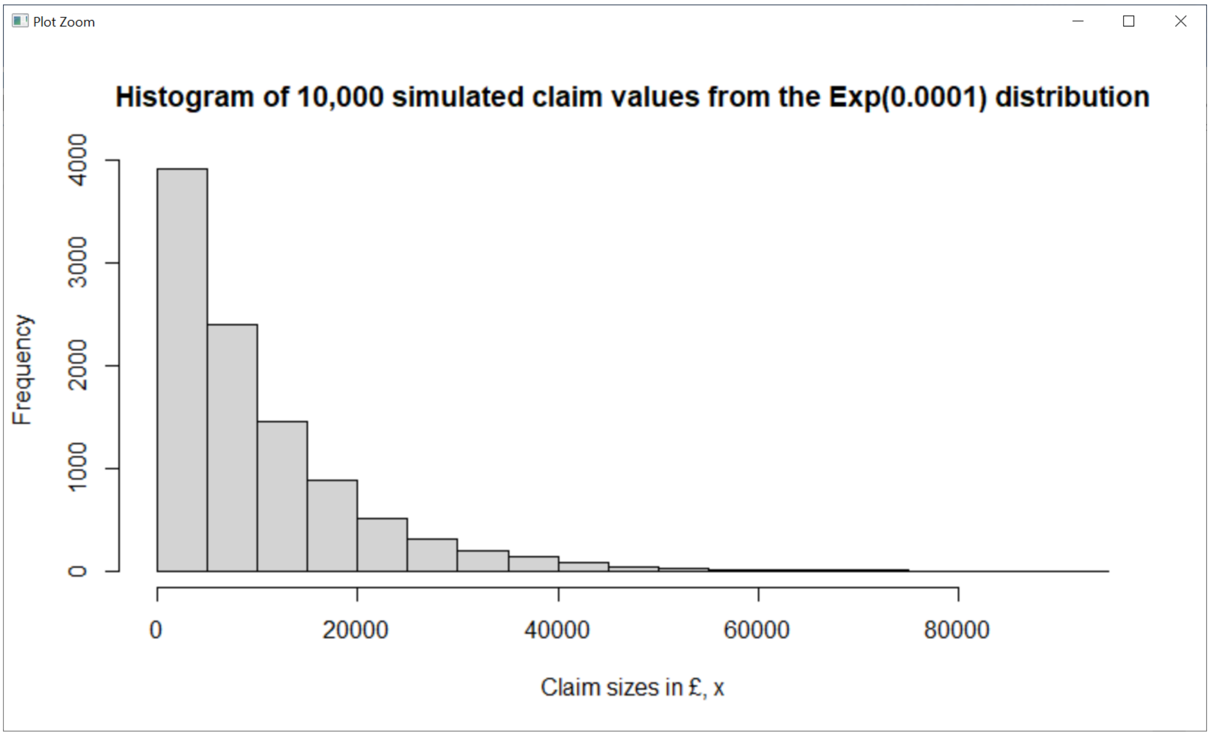
(a) the insurer
(b) the reinsurer

under each of the following six retention levels:
£5,000, £10,000, £20,000, £30,000, £40,000 and £50,000.

(v) Plot your results from part (iv) on four separate charts to show how the mean and variance of the claims paid by the insurer and reinsurer vary with the retention level.

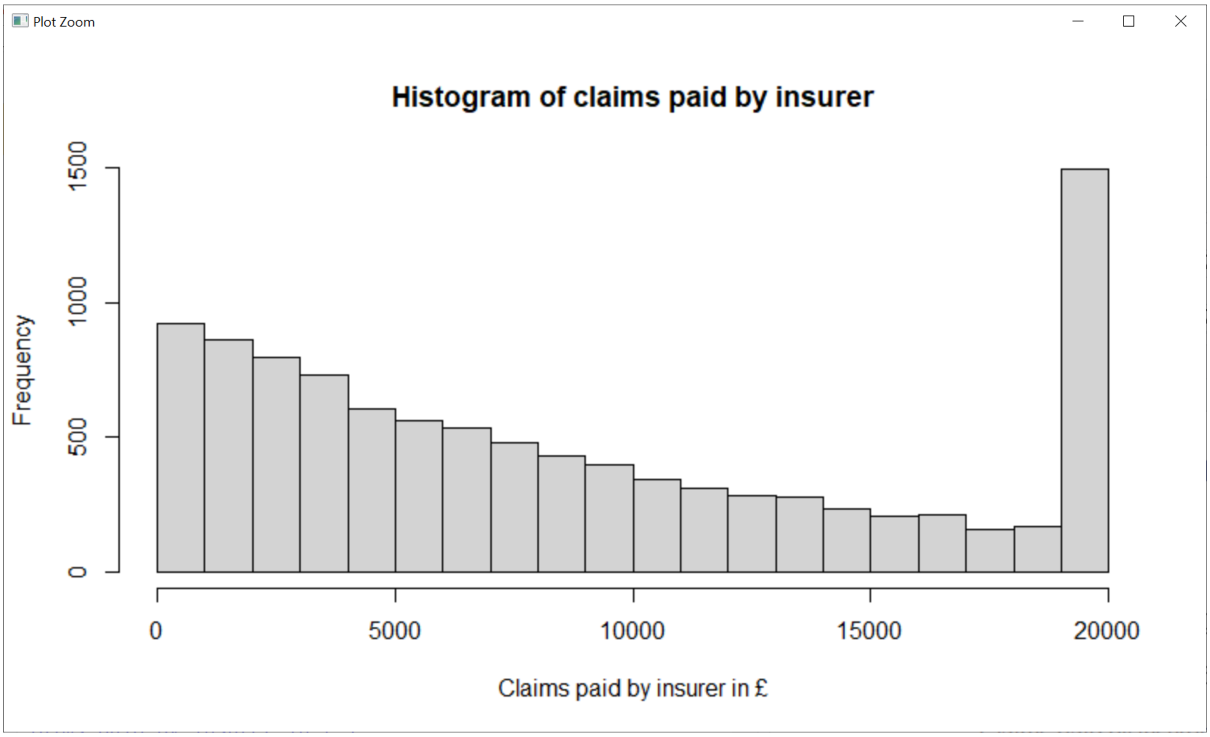
Answer:

```
#(i)Simulate 10000 claims from Exp(0.0001) and plot histogram

set.seed(123)

n<-10000

lambda<-0.0001

x<-rexp(n,lambda)

hist(x,xlab="Claim sizes in £, x",main="Histogram of 10,000
simulated claim values from the Exp(0.0001) distribution")
```




```
#(ii)Mean and var of simulated values

> mean(x)

[1] 10037.81

> var(x)
```

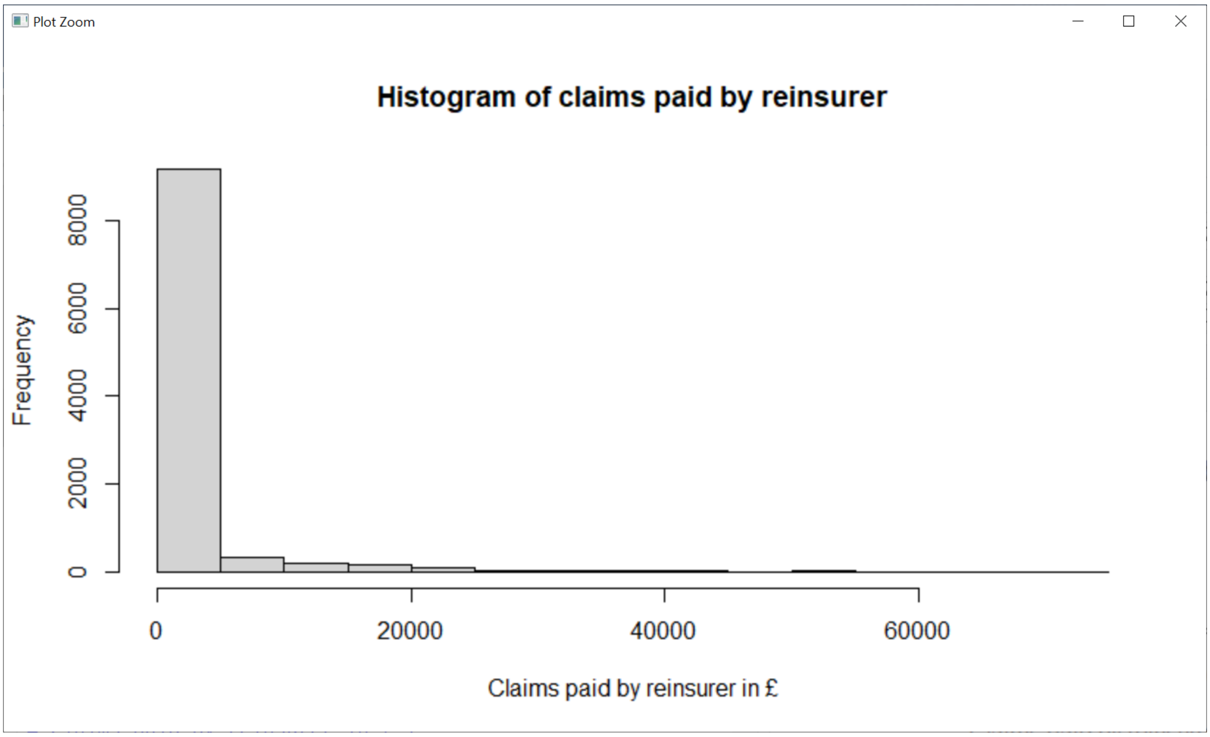
```
[1] 99976440

#(iii)(a)Mean and var of claims paid by insurer. Let Y represent the claims paid by insurer.

y<-pmin(x,20000)

> mean(y)

[1] 8671.328

> var(y)

[1] 43964610

hist(y,main="Histogram of claims paid by insurer",xlab="Claims paid by insurer in £")
```




```
#Q3(iii)(b)Mean and var of claims paid by reinsurer. Let Z represents the claims paid by reinsurer.

z<-x-y

z

> mean(z)

[1] 1366.485

> var(z)

[1] 25047814

> length(z)
```

```
[1] 10000
```

```
hist(z,main="Histogram of claims paid by reinsurer",xlab="Claims
paid by reinsurer in £")
```



ALTERNATIVELY, USE THE FOLLOWING. NOTE THAT WE GET THE SAME RESULTS:

```
> z<-pmax(0,x-20000)

> mean(z)

[1] 1366.485

> var(z)

[1] 25047814

> length(z)

[1] 10000
```

```
#(iv) Different retention levels. Define a vector using the c()
function in R.

M<-c(5000,10000,20000,30000,40000,50000)

M1<-5000

M2<-10000

M3<-20000

M4<-30000

M5<-40000

M6<-50000


#(iv)(a)Mean and var of claims paid by insurer under different
retention limits.

> y1<-pmin(x,M1)

> y2<-pmin(x,M2)

> y3<-pmin(x,M3)

> y4<-pmin(x,M4)

> y5<-pmin(x,M5)

> y6<-pmin(x,M6)

> ins_mean_vector<-
c(mean(y1),mean(y2),mean(y3),mean(y4),mean(y5),mean(y6))

> ins_var_vector<-c(var(y1),var(y2),var(y3),var(y4),var(y5),var(y6))

> ins_mean_vector

[1] 3940.658 6339.937 8671.328 9538.658 9869.412 9971.611

> ins_var_vector

[1]  2533775 12858446 43964610 70204248 86376566 93370367
```

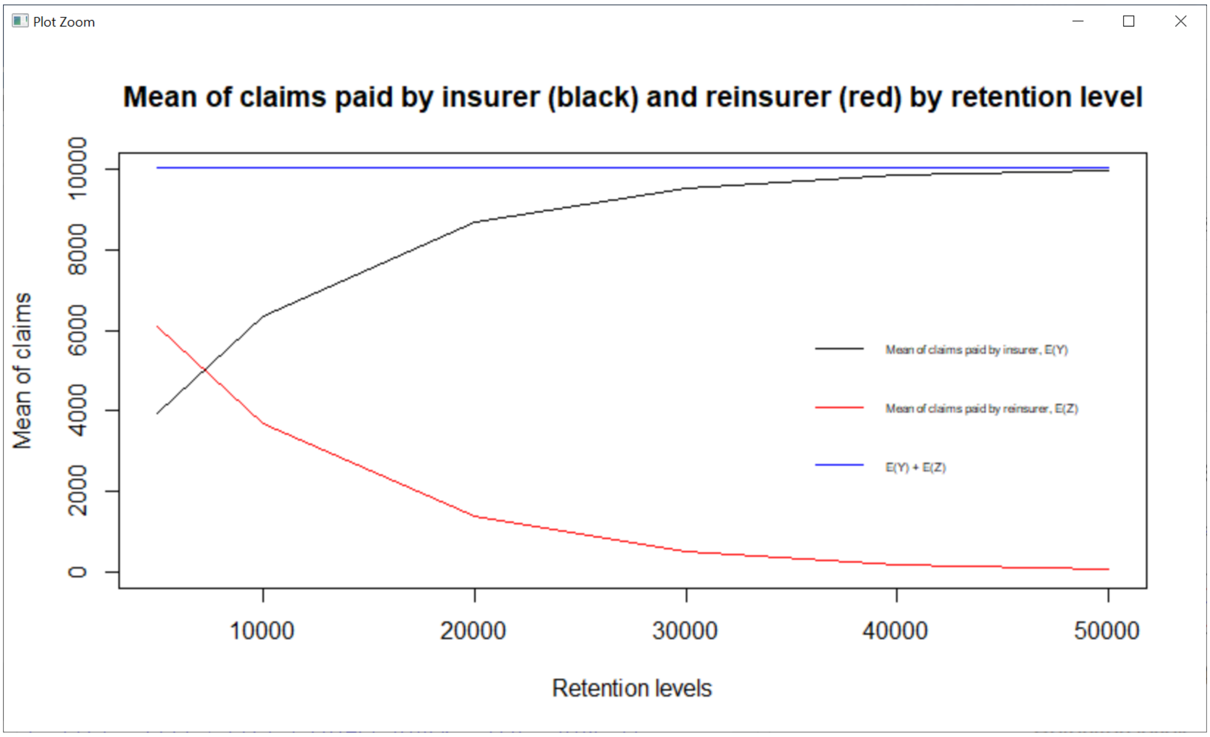
```
>(iv)(b)Mean and var of claims paid by reinsurer under different
retention limits.

> z1<-x-y1

> z2<-x-y2

> z3<-x-y3

> z4<-x-y4

> z5<-x-y5

> z6<-x-y6


> reins_mean_vector<-
c(mean(z1),mean(z2),mean(z3),mean(z4),mean(z5),mean(z6))

> reins_var_vector<-
c(var(z1),var(z2),var(z3),var(z4),var(z5),var(z6))

> reins_mean_vector

[1] 6097.15469 3697.87535 1366.48492  499.15454

[5]  168.40057   66.20174

> reins_var_vector

[1] 84523429 60046376 25047814  9343406  3450843  1305645
```

Note that part (v) asks for 4 separate charts but here, we have

- combined the charts for insurer's mean and reinsurer's mean into one chart,
- and combined the charts for insurer's variance and reinsurer's variance into one chart.

The blue lines in the charts below are not required by the question and have been added for comparison purposes only.

```
#(v)Plot results from (iv)

plot(M,ins_mean_vector, ylim=c(0,10000),type="l",xlab="Retention levels",ylab="Mean of claims",main="Mean of claims paid by insurer (black) and reinsurer (red) by retention level")

lines(M,reins_mean_vector,col="red")

lines(M,ins_mean_vector+reins_mean_vector,col="blue")

legend(35000,7000,cex=0.5, bty="n", lty=1,c("Mean of claims paid by insurer, E(Y)","Mean of claims paid by reinsurer, E(Z)","E(Y) + E(Z)"),col=c("black","red","blue"))
```



```
plot(M,ins_var_vector, type="l",xlab="Retention
levels",ylab="Variance of claims",main="Variance of claims by
retention level")

lines(M,reins_var_vector,col="red")

lines(M,ins_var_vector+reins_var_vector,col="blue")

legend(35000,70000000, cex=0.5, bty="n", lty=1,c("Variance of claims
paid by insurer, var(Y)","Variance of claims paid by
reinsurer,var(Z)","var(Y) + var(Z)"),col=c("black","red","blue"))
```