MTH6102: Bayesian statistical methods
2023
Revision exercise sheet 12

1. Hurricane-strength wind speeds (X miles per hour) for locations in the Gulf of Mexico are modelled using a distribution with probability density function (pdf)

$$f(x \mid \sigma) = \frac{x}{\sigma^2} \exp\{-\frac{x^2}{2\sigma^2}\}.$$

During a hurricane, you collect wind speeds at n randomly chosen locations in the Gulf of Mexico, giving $x_1, \ldots, x_n$. Find the likelihood function for σ and the maximum likelihood estimator (MLE) for σ.

2. A dishonest gambler has a box containing 10 dice which all look the same. However there are actually three types of dice.

- There are 6 dice of type A which are fair dice with 1/6 probability of getting a six.
- There are 2 dice of type B which are biased with 0.8 probability of getting a six.
- There are 2 dice of type C which are biased with 0.04 probability of getting a six.

The gambler takes a die from the box at random and rolls it and gives 6. Find the posterior probability for each type of die.

3. Suppose that a random sample $x_1, \ldots, x_n$ is obtained from $f(x \mid \theta)$. Derive the posterior distributions for the following models:

(a) $f(x \mid \theta) = \theta^{x-1}(1-\theta)$, $x = 1, 2, 3, \ldots$ with a $U(0, 1)$ prior distribution for θ.

(b) $f(x \mid \theta) = \frac{e^{-\theta}\theta^x}{x!}$ with Exponential(1) prior distribution for θ.

4. Consider an experiment with a possibly loaded six-sided die. Let $\theta = P(\text{rolling a six})$. Before conducting the experiment, we believe that all values of θ are equally likely and so we use a $U(0, 1)$ random variable as the prior distribution for θ. We then roll the die 10 times and observe 3 sixes.

(a) Obtain the likelihood function and use it to obtain the posterior distribution, $p(\theta \mid x = 3)$.

(b) What is the posterior mean of θ?

5. The proportion θ of defective items in a large shipment is unknown. However, experience suggests that a beta$(2, 200)$ prior is appropriate.

(a) Suppose that 100 items are selected at random from the shipment and that three are found to be defective. What is the posterior distribution of θ?

(b) Suppose that another statistician, having observed the three defectives, said that his/her posterior distribution for θ was a beta distribution with mean 4/102 and variance. 0.0003658. What prior distribution had that statistician used?

(c) A third statistician suggests to use a beta(a, b) prior distribution such that the prior mean is 0.4 and the prior standard deviation is 0.2. What values of a and b should he/she choose?

6. A random sample of size n is to be taken from a $N(\theta, 2^2)$ distribution. The prior for θ is $N(\mu, 1/d)$.

(a) For an uninformative prior, do we need a large or small value for d? If we choose a small value for d, what is the effect on the posterior mean for θ, compared to a large d?

(b) If $d = 1$ what is the smallest number of observations that must be included in the sample if the standard deviation of the posterior distribution of θ is to be reduced to 0.1?

(c) If $n = 100$, show that no matter how large the value of d, the standard deviation of the posterior distribution is less than 1/5.

7. Suppose we have a simple random sample from a Poisson distribution, that is $x_i \sim \text{Poisson}(\theta)$, $i = 1, \ldots, n$

(a) Derive the Jeffreys prior for the Poisson likelihood.

(b) Is the Jeffreys prior a proper prior distribution? Explain your answer

(c) Derive the posterior distribution assuming Jeffreys prior for θ.

Suppose now we use a Gamma distribution, Gamma(a, b) as a prior for θ where $a > 0$ and $b > 0$ are known.

(d) Why might it be convenient to assume a Gamma prior distribution for θ?

(e) Derive the posterior distribution for θ under the Poisson model and the prior Gamma(a, b) distribution.

(f) Suppose $a = b$. For an uninformative prior, do we need a large or small value for a? If we choose a small value for a, what is the effect on the posterior distribution for θ compared to a large a?

8. The Pareto distribution is often used to model data in many areas, ranging from the wealth of individuals to the sizes of meteorites. Suppose $x = (x_1, \ldots, x_n)$ is a random sample from a Pareto distribution Pa$(1, \theta)$ with pdf

$$p(x_i \mid \theta) = \frac{\theta}{x_i^{\theta+1}}, \quad x \geq 1.$$

(a) What is the likelihood function $p(x \mid \theta)$?

Suppose your prior beliefs about θ were described by a Gamma(a, b) distribution where both $a > 0$ and $b > 0$ are known.

(b) Determine your posterior density for θ. Name this distribution, including its parameters.

(c) Is the Gamma distribution a conjugate prior distribution in this case? Explain your answer.

(d) Let the last three digits of your ID number be ABC. Suppose we want the prior mean for θ to be 5 + A and the prior standard deviation to be 5 + B. Find the prior distribution parameters that satisfy this.

(e) Using thise prior distribution from (d), find the posterior distribution for θ. What is the posterior mean?

9. Let y be the number of successes in n independent Bernoulli trials and let θ be the probability of success. Suppose your prior beliefs about θ are described by a Beta(a, b) distribution where both $a > 0$ and $b > 0$ are known.

(a) Find the posterior predictive probability of observing a new success x on the next Bernoulli trial.
Let x be the number of future successes on the next m independent Bernoulli trials

(b) Find the posterior predictive probability of x.

(c) Using the law of iterated expectation find the mean of x.

(d) Using the law of total variance find the variance of x.

10. The distribution of flaws along the length of an artificial fibre follows a Poisson process, and the number of flaws in a length ℓ is Poisson$(\ell_i \theta)$. The number of flaws in five fibres of lengths 10, 15, 25, 30 and 40 metres were found to be 3, 2, 7, 6, 10 respectively. We are interested in finding the predictive distribution for the number of flaws in another piece of length 60 metres. The conjugate prior for the Poisson model is $\theta \sim$ Gamma(a, b) distribution where both $a > 0$ and $b > 0$ are known.

(a) Determine the posterior distribution for θ.

(b) Determine the predictive distribution for the number of flaws Y in another piece of length ℓ.

(c) Use the information in the data to determine the number of flaws in another piece of length 60 metres.