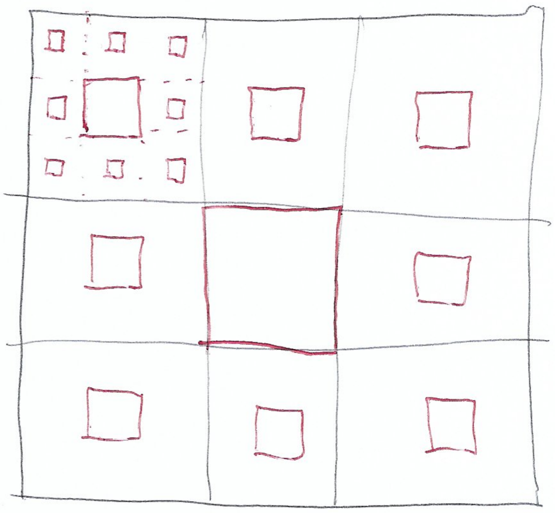
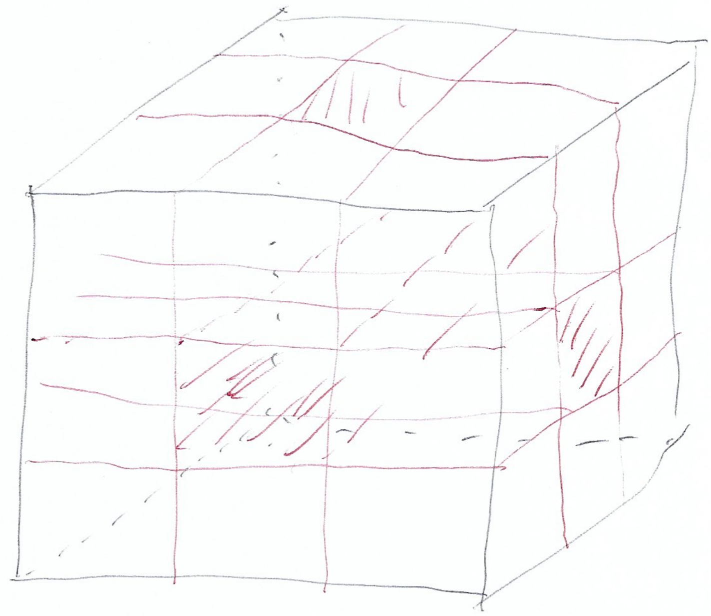
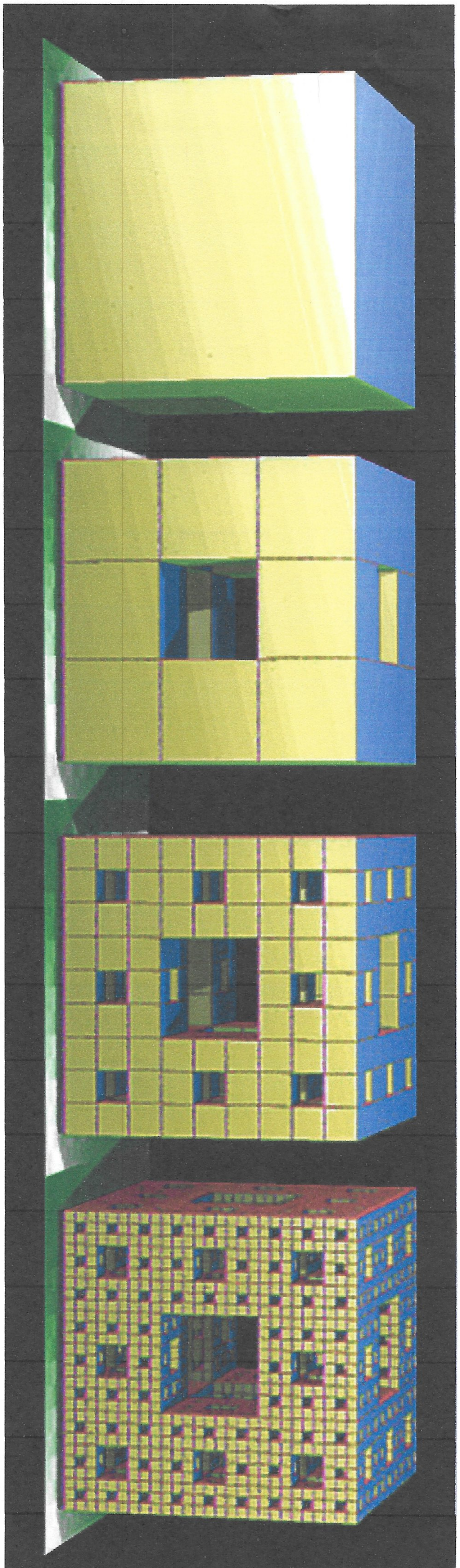
More examples

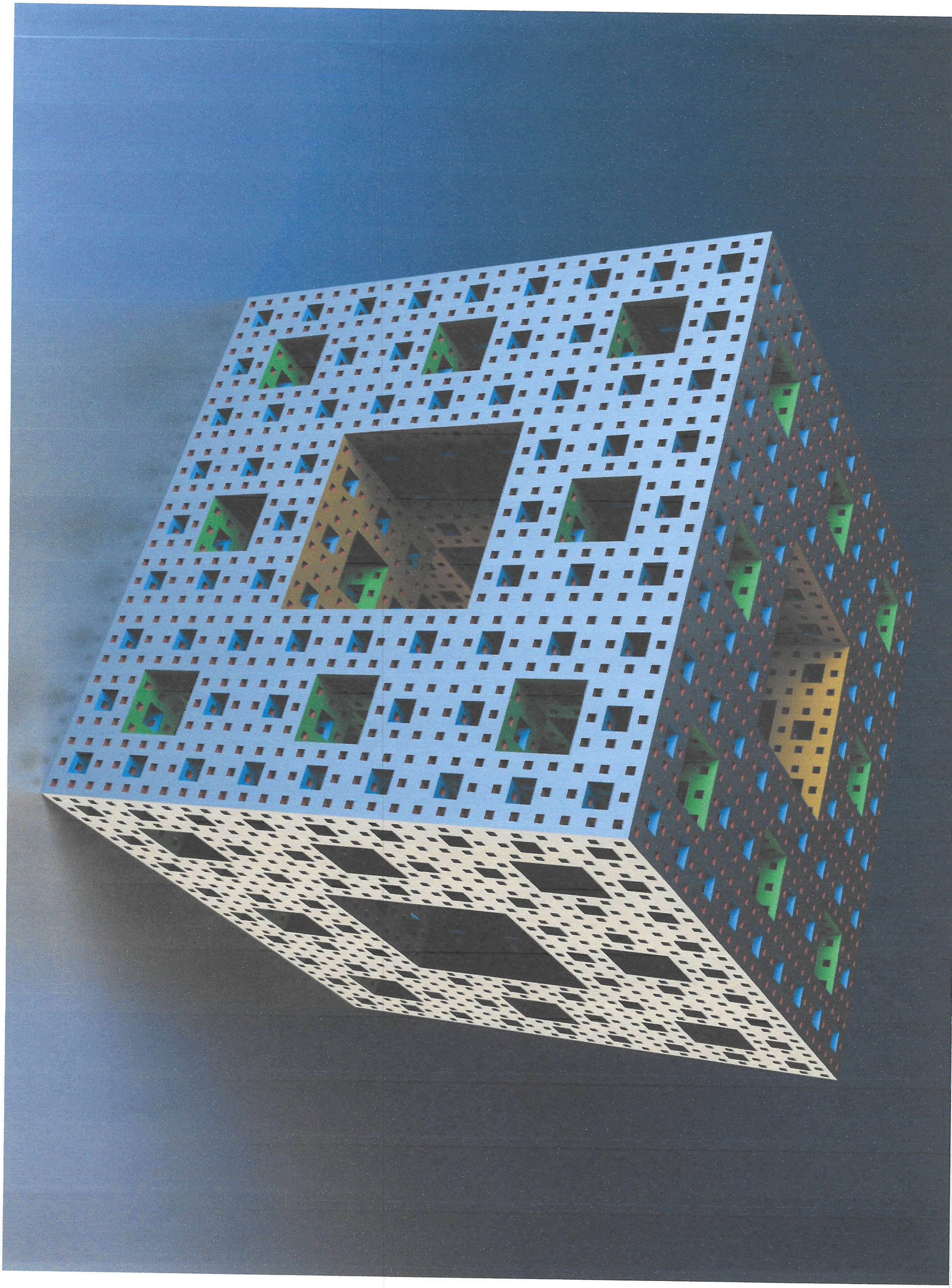
Sierpinski carpet



Menger sponge





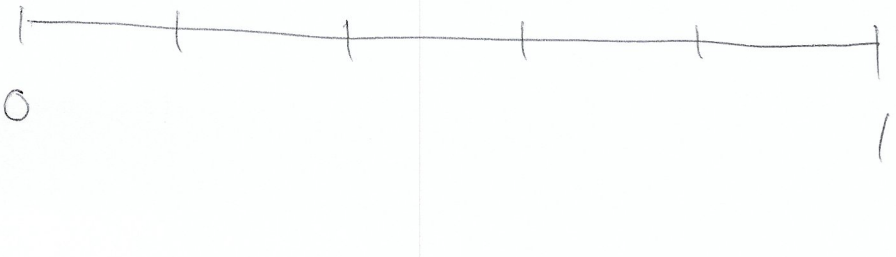


Dimension

There are various ways of thinking about the 'dimension' of a geometric object A $(\subseteq \mathbb{R}^d)$

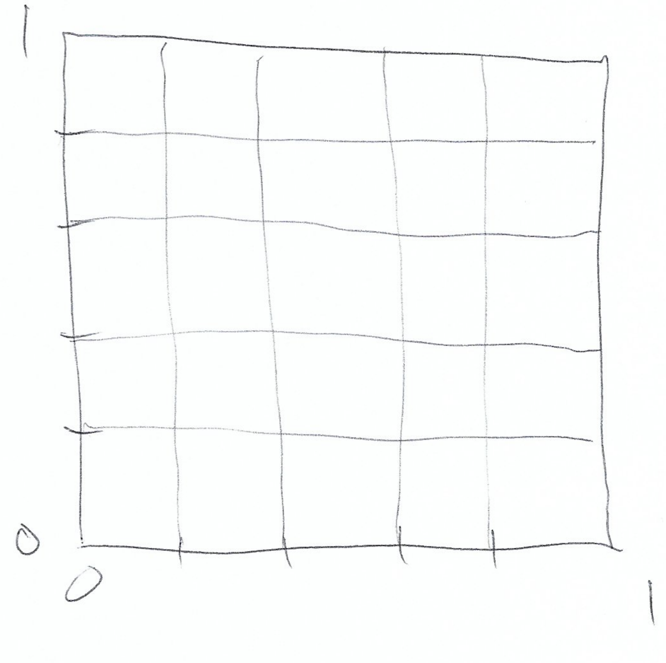
A fruitful approach stems from the following simple observation:

For $A = [0,1] \subset \mathbb{R}$, if we divide A up into sub-intervals of length $\varepsilon = 1/k$ then there are $k = k^1$ such intervals.

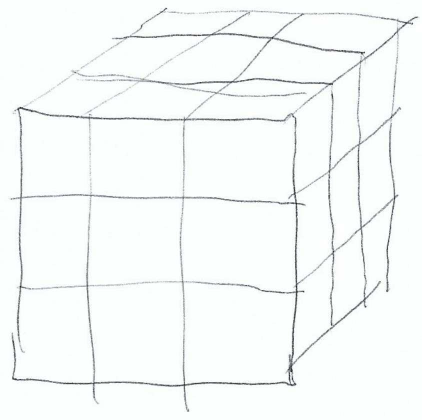


For $A = [0,1]^2 = \{(x,y) : x,y \in [0,1]\}$
$\subseteq \mathbb{R}^2$

If we divide A up into sub-squares of side length $\varepsilon = 1/k$ then there are k^2 such sub-squares.



For $A = [0,1]^3 \subset \mathbb{R}^3$ (the unit cube), if we divide up into sub-cubes of side length $\varepsilon = 1/k$ then there are k^3 such sub-cubes



For $A = [0,1]^d = \underbrace{[0,1] \times [0,1] \times \ldots \times [0,1]}_{d \text{ times}}$,

let $N(\varepsilon) = N(1/k) = k^d$ be the number of "sub-cubes" of side length $\varepsilon = 1/k$ needed to fill A.

Intuitively, $A = [0,1]^d$ is a d-dimensional object, and the value d can be extracted / recovered by considering the quantity

$$\frac{\log N(\varepsilon)}{\log(1/\varepsilon)} = \frac{\log(k^d)}{\log k}$$

$$= \frac{d \log k}{\log k} = d$$

<u>Idea</u> Extend this thinking to define dimension of more general sets:

Box dimension (or 'box counting dimension')

For $d \in \mathbb{N}$, a ~~d~~ <u>d-dimensional cube</u> (or 'box') of side length ε is a subset of $\mathbb{R}^d$ of the form

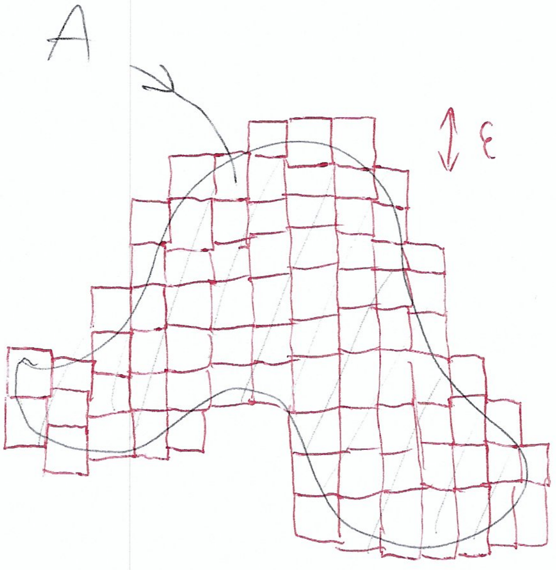
$$[a_1, a_1+\varepsilon] \times [a_2, a_2+\varepsilon] \times \ldots \times [a_d, a_d+\varepsilon]$$

(Such cubes have volume ε^d)

Let A be a subset of $\mathbb{R}^d$.

Let $N(\varepsilon)$ denote the smallest number of cubes of side length ε needed to <u>cover</u> A.

This means that the union of these cubes is larger than A, i.e. the union contains A as a subset.



<u>Defn</u> The <u>box dimension</u> (or <u>box counting dimension</u>) of A is given by

$$D(A) := \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)}$$

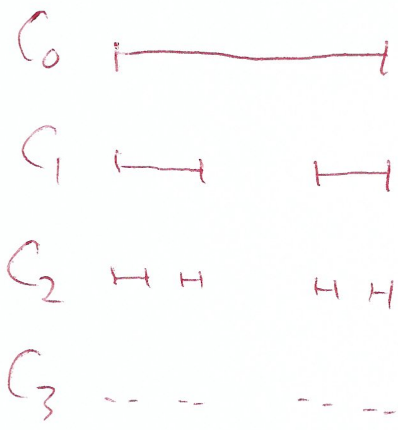
$$= \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{-\log \varepsilon}$$

(if this limits exists)

<u>Idea</u>: $N(\varepsilon) \sim \left(1/\varepsilon\right)^{D(A)}$

<u>Defn</u> The set A is called a <u>fractal</u> if $D(A)$ is not an integer.

Middle-$\frac{1}{3}$ Cantor set



$$C = \bigcap_{n=0}^{\infty} C_n$$

If $\varepsilon = \frac{1}{k} = \frac{1}{3^n}$ then C is a subset of each C_n.

i.e. The collection of length-$\frac{1}{3^n}$ intervals making up C_n covers C, in the sense that their union contains C.

Since C_n consists of 2^n intervals, each of length $\frac{1}{3^n} = \varepsilon$, then $N(\varepsilon) = 2^n$.

So
$$\frac{\log N(\varepsilon)}{\log(1/\varepsilon)} = \frac{\log 2^n}{\log 3^n} = \frac{n \log 2}{n \log 3}$$
$$= \frac{\log 2}{\log 3} \simeq 0.631\ldots$$

The middle-$\frac{1}{3}$ Cantor set C has box dimension equal to $\frac{\log 2}{\log 3}$, hence it is a fractal.

<u>Example</u> (cf. Example of the set K of numbers in $[0,1]$ with decimal expansions only using digits $\{1, 4, 9\}$)

Let $\varepsilon = 1/k = \frac{1}{10^n}$

Recall $K = \bigcap_{n=0}^{\infty} \Phi^n([0,1])$, and note that

$$\Phi^0([0,1]) \supset \Phi^1([0,1]) \supset \Phi^2([0,1]) \supset \ldots$$

So $K \subset \Phi^n([0,1])$ for all $n \geq 0$.

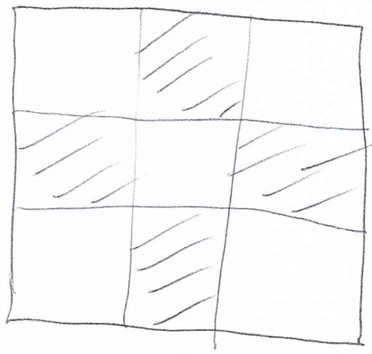
Note that $\Phi^n([0,1])$ has 3^n disjoint closed intervals, each of length $1/10^n$.

Then $\dfrac{\log N(\varepsilon)}{\log(1/\varepsilon)} = \dfrac{\log 3^n}{\log 10^n} = \dfrac{n \log 3}{n \log 10}$

$$= \frac{\log 3}{\log 10} \simeq 0.4771\ldots$$

This is the box dimension of K, so K is a fractal.

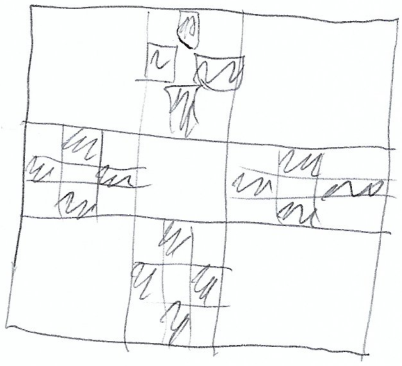
Example ('Checkerboard' in $\mathbb{R}^2$)



F

Let $\varepsilon = 1/k = 1/3^n$

Each $F_n = \Phi^n([0,1]^2)$ is the union of 4^n squares, each of side length $1/3^n$.



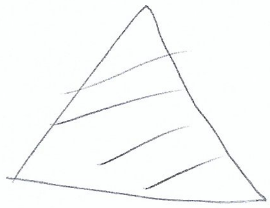
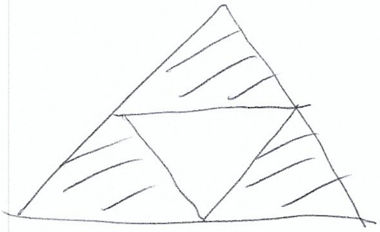
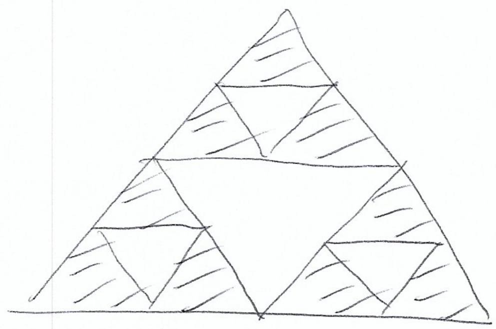
F_2

Then
$$\frac{\log N(\varepsilon)}{\log(1/\varepsilon)} = \frac{\log 4^n}{\log 3^n} = \frac{n\log 4}{n \log 3}$$
$$= \frac{\log 4}{\log 3} \simeq 1.26\ldots$$

$\left(= 2 \times \frac{\log 2}{\log 3} = 2 \times \text{The box dimension of the middle-}1/3 \text{ Cantor set } C \right)$

The box dimension of F is $\frac{\log 4}{\log 3}$, and since this is not an integer then F is a fractal.

The Sierpinski Triangle

n	P_n
0	
1	
2	
	$\vdots$

The Sierpinski triangle is defined to be $P = \bigcap_{n=0}^{\infty} P_n$

Let us express the Sierpinski triangle in the language of iterated function systems.

For $j = 1, 2, 3,$

let $z_j^* \in \mathbb{R}^2 \equiv \mathbb{C}$ — This identification is purely for convenience of notation

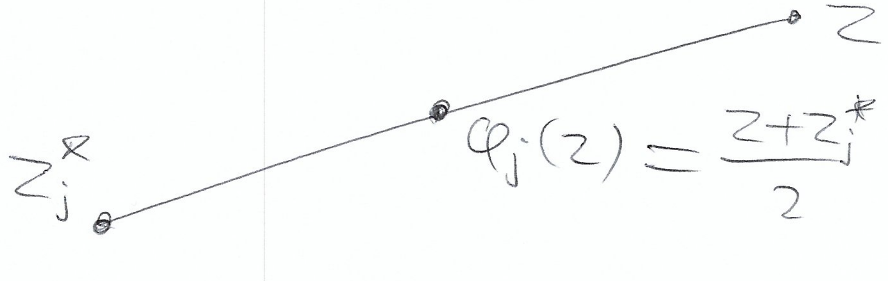
be three non-collinear,

and define

$$\varphi_j : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ \parallel\!| & & \parallel\!| \\ \mathbb{C} & & \mathbb{C} \end{array}$$

by

$$\varphi_j(z) = \frac{z + z_j^*}{2} = \frac{z}{2} + \frac{z_j^*}{2}$$



So $\varphi_j(z)$ is the mid-point between z and z_j^*.

Clearly $\varphi_j(z_j^*) = z_j^*$

i.e. z_j^* is a fixed point of φ_j

(in fact it is the unique fixed point)

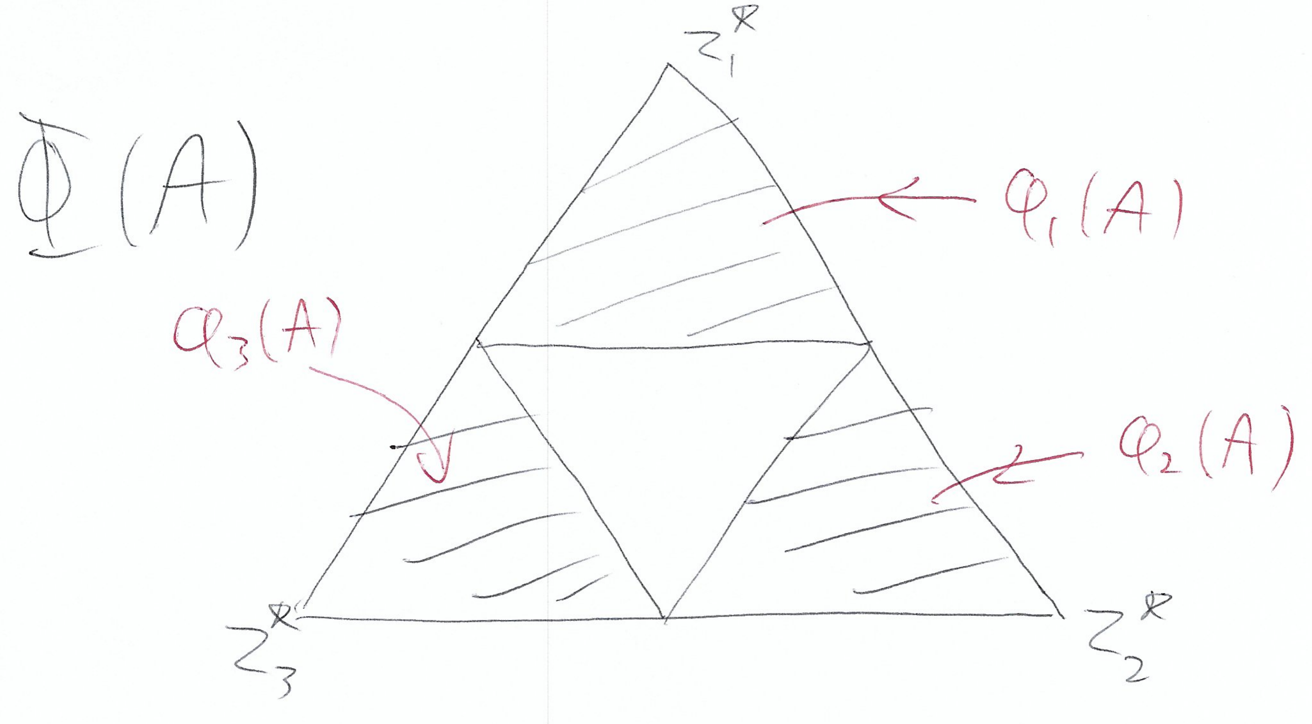
Moreover, z_j^* is an attracting fixed point for φ_j, and its basin of attraction is the whole of $\mathbb{R}^2 \equiv \mathbb{C}$.

The maps $\varphi_1, \varphi_2, \varphi_3$ ~~const~~ constitute an iterated function system, and we can (as usual) associate the map Φ defined by

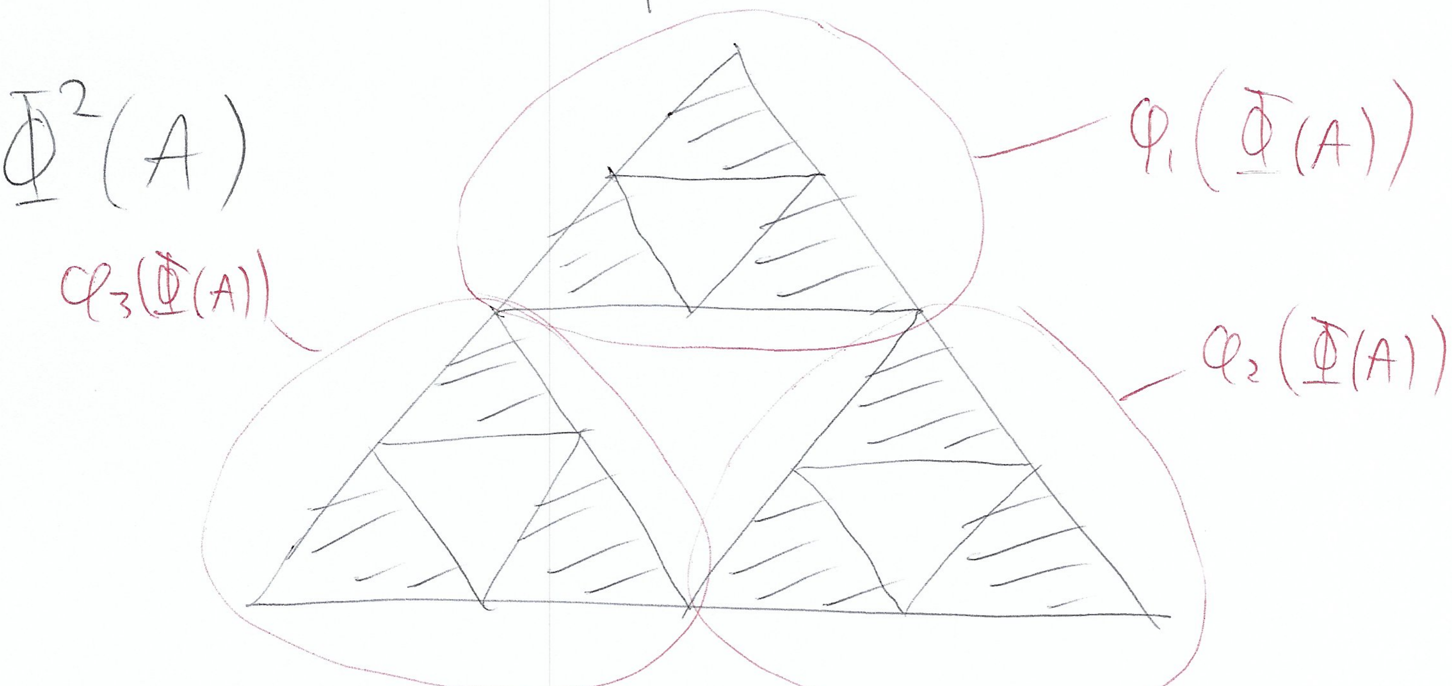
$$\Phi(A) = \bigcup_{j=1}^{3} \varphi_j(A)$$

for all $A \subset \mathbb{R}^2 \equiv \mathbb{C}$

~~Now~~ If for example A is the solid triangle with vertices z_1^*, z_2^*, z_3^* then:



$$\downarrow \Phi$$



In general, $P_n = \Phi^n(A)$, and the Sierpinski triangle P is

$$P = \bigcap_{n=0}^{\infty} \Phi^n(A).$$