MTH6157 Survival Models

January 2023 Solutions

1. Survival Model principles
   (a) advantages and disadvantages:
   - simple model, adapted from exponential model
   - captures nature of rising force of mortality with age
   - can chain together exponential survival probabilities
   - will be inaccurate, especially where force changing fast (over age 50)
   - possible to increase accuracy without too much more complexity

   (b) using the exponential model with constant force of mortality

   p_66 = exp(-mu_66) = exp(-0.008) = 0.992032

   (c) for the 67 – 28 = 39 year survival probability we can chain together the exponential model survival probabilities at different ages

   39_p_28 = exp(-2x0.001).exp(-10x0.002).exp(-10x0.003).exp(-10x0.005).exp(-7x0.008)

   = 0.85385

   (d) Pr(life assurance claim) = Pr(death before age 67) = 23_q_44 = 1 – 23_p_44

   = 1 – exp(-6*0.003).exp(-10*0.005).exp(-7*0.008)

   = 1 – 0.88338 = 0.11662

IFoA syllabus 4.1.3 and 4.1.4

part (a) adapted from lecture, (b)(c)(d) similar to exercise sheet but more challenging

2. Kaplan Meier estimate
   (a) after j days let

   n_j = number of books still out on loan

   d_j = number of books returned on day j

   c_j = number of books censored on day j

   λ_j = the hazard of returning a book at day j = d_j / n_j

   S(j) = $\prod_{i \le j}(1 - \lambda_i)$

j	n	d	c	λ	1-λ	S(j)
1	178	25	0	0.140449	0.859551	0.859551
2	153	28	0	0.183007	0.816993	0.702247
3	125	49	1	0.392	0.608	0.426966
4	75	39	0	0.52	0.48	0.204944
5	36	22	14	0.611111	0.388889	0.079700
6	0					

The final S(j) column is the K-M estimate of the survival function

(b) we need highted j such that S(j) < 1 – 0.8 = 0.2 hence j = 5 days here

(c) might be concerned by:

- the 80% threshold is very nearly met after j=4 days
- in fact with just one more book returned at j=4 then S(4) would be < 0.2
- this is a small sample compared to the whole library system
- the number of books < number of students borrowing so the observations are not independent
- the pattern for these Politics students might not be replicated for other students
- note that censoring does not seem to be an issue here

IFoA syllabus 4.2.3

part (a) (b) similar seminar, part (c) unseen and higher order

3. Cox's Proportional Hazard

(a) selection present here:

- time selection – the problem of the out of date mortality table
- class selection – smokers and non-smokers
- spurious selection – genuine random variation that might appear as one of the 2 above
- the risk of adverse selection from non-smokers selecting this insurance provider
- *note for full marks need both the name of the selection and comment on why it is relevant*

(b) the baseline hazard is the policyholder with all $z_i = 0$

so here that is a 60-year-old non-smoker with a pensions annuity

(c) here $z_1 = 1$, $z_2 = 14$, $z_3 = 0$

so beta * z^T = 0.423 + 14*0.62 = 9.103

so $\lambda_t = \lambda_0 * \exp(9.103)$

and S(t) = $\exp[-\int_0^t \lambda_0 \exp(9.013)\, dt]$

(d) we know that for the 74-year-old S(2) = = $\exp[-\int_0^2 \lambda_0 \exp(9.013)\, dt] = 0.98$

therefore $\exp[-\int_0^t \lambda_0 dt] = 0.98^{\exp(-9.013)}$

so for the 66-year-old $z_1 = 0$, $z_2 = 6$, $z_3 = 1$

and beta*Z^T = 6*0.62 – 0.3 = 3.42

and their 2 year survival probability is

S(2) = $\exp[-\int_0^2 \lambda_0 \exp(3.42)\, dt]$

$= (0.98^{\exp(-9.013)})^{\exp(3.42)}$

= 0.999931

(e) the null hypothesis is that the Cox's PH model with the addition of the policy-type z_3 covariate has no more explanatory power for the hazard than a 2 covariate Cox model with age and smoker status

- we would fit both the 2 and 3 covariate Cox models
- and calculate the maximised log likelihood for these log L_2 and log L_3
- the likelihood ratio statistic = -2(log L_2 - log L_3)
- which has a chi-squared distribution on 1 degree of freedom
- so we reject the null hypothesis at the 95% significance level if the likelihood ratio statistic is greater than $\chi^2_{0.95;1}$

IFoA syllabus 4.2.1 and 4.2.6

part (a) unseen, (b)(c)(d) similar seminar exercise, part (e) adapted from lecture

4. Graduation methods

(a) advantages are that smoothness should be guaranteed

if the previous table was smoothed

the graduation is applicable over the full age range

including very young and old

but COVID may well have changed the shape of the distribution

especially at older ages

and other class selection or time selection changes may be surpressed

heterogeneity in the data that led to the previous table is still an issue

(b) to choose a function we would first plot the new investigation q against the old table q

or plotting -log(1-q) better if we looking for relationship in mu

function (1) more appropriate if mortality higher or lower across all ages

function (2) if shape of mortality curve shifted

then two ways of fitting a or b

maximum likelihood or least squares

(c) splines split the 0 to 110 age range into different parts

then fits parametric formula to each part of age range

this would allow for COVID having different affects at different age ranges

although selection of ages for knots is difficult

would also allow for other mortality changes that vary by age e.g. medical advances, class selection

IFoA syllabus 4.5.4

part (a) unseen application of lecture, higher order, (b) lecture (c) unseen, higher order

5. Graduation statistical tests
   (a) this is the chi squared test

H0: the graduated rates represent the true underlying mortality

H1: the graduated rates do not represent the true underlying mortality

we calculate standardised deviations at each age, z_x

Observed deaths, O = exposed to risk * model estimate mu

Expected deaths, E = exposed to risk * graduated rate mu

then z_x = $\frac{O-E}{\sqrt{E}}$

and our test statistic is X = $\sum z_x^2$

which follows a χ^2 distribution

where degrees of freedom = 12 age groups – 1 (choice of standard table) – 1 (parameter fitted) = 10

note will accept other degrees of freedom less than 12 (but not 12) as long as explanation given

calculations are:

age x	exposed to risk	model estimate	graduated rate	O	E	z_x	z_x^2
71	2245	0.0142	0.0138	31.879	30.981	0.161335	0.026029
72	2134	0.0154	0.0145	32.8636	30.943	0.345268	0.11921
73	2045	0.0156	0.0158	31.902	32.311	-0.07195	0.005177
74	2004	0.0169	0.0165	33.8676	33.066	0.139401	0.019433
75	1945	0.0195	0.0175	37.9275	34.0375	0.666762	0.444571
76	1904	0.0204	0.0196	38.8416	37.3184	0.249342	0.062171
77	1834	0.0215	0.0213	39.431	39.0642	0.058687	0.003444
78	1783	0.0236	0.0227	42.0788	40.4741	0.252235	0.063622
79	1728	0.0268	0.0248	46.3104	42.8544	0.52793	0.27871
80	1649	0.0296	0.0286	48.8104	47.1614	0.240119	0.057657
81	1622	0.0328	0.0326	53.2016	52.8772	0.044611	0.00199
82	1594	0.0359	0.0357	57.2246	56.9058	0.042261	0.001786
sum							1.083801

X = 1.083801 < χ^2(0.95,10) = 3.94

therefore we do not reject the null hypothesis

on the basis of this test the graduated rates are a good representation of the model output

   (b) Cumulative deviations test and Signs test

(c) the cumulative deviations test

H0: the graduated rates represent the true underlying mortality

the test statistic is

$$X = \frac{\sum O - \sum E}{\sqrt{\sum E}} \sim N(0,1)$$

here $\sum O$ = 494.3381, $\sum E$ = 477.994 so X = 0.747566 < 1.96

therefore we do not reject the null hypothesis

on the basis of this test the graduated rates are a good representation of the model output

the signs test

H0: the graduated rates represent the true underlying mortality

from the table in (a) we have 11 positive and 1 negative z_x

Probability of this under H0 is $\binom{12}{0}\frac{1}{2}^{12} + \binom{12}{1}\frac{1}{2}^{12}$ = 13 * $½^{12}$ = 0.003174 < 0.025

therefore we reject H0 at 95% level

there is evidence that the graduated rates consistently underestimate the true mortality

(d) we see that the overall fit is good from (a)

so we seek further evidence via the two tests in (c)

the cumulative deviations test suggest that there is no large overall bias at these ages

the signs test however suggests that there is consistent bias

11 of 12 deviations of same sign is statistically significant

which overall suggests there is evidence to support the claim that mortality is underestimated at these ages

which may or may not be due to COVID

but that the size of the underestimate is not large

however best practice would be to look again at the graduation method

IFoA syllabus 4.5.1, 4.5.5, 4.5.7

part (a) similar seminar (b) lecture (c) similar seminar (d) unseen, higher order, challenging