Queen Mary
University of London

Main Examination period 2023 – January – Semester A

MTH5104: Convergence and Continuity

Duration: 2 hours

The exam is intended to be completed within **2 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

All work should be **handwritten** and should **include your student number**. Only one attempt is allowed – **once you have submitted your work, it is final**.

In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;

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Question 1 [25 marks].

(a) Prove that the equation

$$x^4 = x + 1$$

does not have any rational solution. [**10**]

(b) Let

$$A = \left\{ \frac{n^2 - 1}{n^3 - 1} : n \in \mathbb{N}, n \neq 1 \right\}.$$

(i) Prove that $A \subseteq \mathbb{R}$ is bounded. [**5**]

(ii) Prove that $\inf A = 0$. Can you replace inf with min? Justify your answer. [**5**]

(iii) Let B be a bounded from above subset of $\mathbb{R}$. Prove that $-B + A$ is bounded from below. [**5**]

Solution: (**Similar seen in class**)

(a) We begin by showing that this equation does not have any integer solutions. Assume that $x = 2k$, $k \in \mathbb{Z}$ is a solution. Then

$$(2k)^4 = 2k + 1.$$

This is not possible because the left-hand side is even and the right-hand side is odd. We reach the same conclusion if $x = 2k + 1$, with $k \in \mathbb{Z}$, is a solution. Indeed,

$$(2k + 1)^4 = (4k^2 + 4k + 1)^2$$

is odd while

$$(2k + 1) + 1 = 2k + 2$$

is even. We can now consider the case that $x = \frac{a}{b}$, where $a \in \mathbb{Z}$, $b \in \mathbb{N}$ with $b \neq 1$ is a solution. We also assume that the highest common factor between a and b is 1. If $x = \frac{a}{b}$ is a rational non integer solution then

$$\frac{a^4}{b^4} = \frac{a}{b} + 1.$$

Multiplying both sides by b we get

$$\frac{a^4}{b^3} = a + b.$$

This leads to a contradiction because the right-hand side is integer and the left-hand side is not integer. So, our equation cannot have a rational solution. [**10**]

(b) Let

$$A = \left\{ \frac{n^2 - 1}{n^3 - 1} : n \in \mathbb{N}, n \neq 1 \right\}.$$

(i) Prove that $A \subseteq \mathbb{R}$ is bounded.

Solution (**Similar seen in class**)

We begin by observing that

$$\frac{n^2-1}{n^3-1} = \frac{(n-1)(n+1)}{(n-1)(n^2+n+1)} = \frac{n+1}{n^2+n+1}$$

Since $n+1 \leq n^2+n+1$ we conclude that

$$0 \leq \frac{n^2-1}{n^3-1} \leq 1.$$

This shows that the set A is bounded. [**5**]

(ii) Prove that $\inf A = 0$. Can you replace inf with min? Justify your answer.

Solution (**Similar seen in class**)

It is clear from the previous inequality that 0 is a bound from below for the set A. We need to prove that for all $\varepsilon > 0$ there exists $n \in \mathbb{N}$ ($n \neq 1$) such that

$$\frac{n+1}{n^2+n+1} < \varepsilon.$$

Since

$$\lim_{n\to\infty} \frac{n+1}{n^2+n+1} = \lim_{n\to\infty} \frac{n}{n^2} \frac{1+\frac{1}{n}}{1+\frac{1}{n}+\frac{1}{n^2}} = 0,$$

then for n large enough we have that

$$\frac{n+1}{n^2+n+1} < \varepsilon,$$

as desired. All the elements of the set A are strictly greater than 0 so we cannot replace inf with min. [**5**]

(iii) Let B be a bounded from above subset of $\mathbb{R}$. Prove that $-B + A$ is bounded from below.

Solution (**Unseen**)

Since B is bounded from above there exists $c \in \mathbb{R}$ such that $x \leq c$ for all $x \in B$. This means that $-x \geq -c$ for all $x \in B$ and proves that $-B$ is bounded from below. Since A is bounded from below as well we have that

$$-x + y \geq -c + 0,$$

for all $x \in B$ and $y \in A$. This shows that $-B + A$ is bounded from below. [**5**]

Question 2 [25 marks].

(a) Let (a_n) be an increasing sequence of real numbers bounded from above and (b_n) be a decreasing sequence.

(i) Is $(a_n b_n)$ necessarily convergent? Justify your answer. [**5**]

(ii) Prove that if in addition $b_n \geq 1$ for all $n \in \mathbb{N}$ then the sequence $(a_n b_n^{-1})$ is convergent. [**5**]

(b) Let (a_n) be the sequence defined recursively by

$$a_1 = \sqrt{3},$$
$$a_n = \sqrt{2a_{n-1} + 3}, \qquad n \geq 2.$$

(i) Prove that (a_n) is bounded. [**5**]

(ii) Prove that (a_n) is increasing. [**5**]

(iii) Making use of (i) and (ii) prove that the sequence (a_n) is convergent and compute its limit. [**5**]

(a) (i) **Solution**: (**Unseen**). This is in general not true. Indeed, the sequence

$$a_n = 1 - \frac{1}{n}$$

is increasing and bounded above and the sequence

$$b_n = -n$$

is decreasing. However,

$$a_n b_n = \left(1 - \frac{1}{n}\right)(-n) = -n + 1$$

is not convergent because it diverges to $-\infty$. [**5**]

(ii) **Solution**: (**Unseen**) By assumptions the sequence (b_n) is decreasing and bounded from below. Therefore (b_n^{-1}) is increasing and bounded above. Both the sequences (a_n) and (b_n^{-1}) are increasing and bounded above so by a result seen in class they are both convergent. Since the limit of a product is the product of the limits we conclude that the sequence $(a_n b_n^{-1})$ is convergent too. [**5**]

(b) **Solution**: (**Similar seen in class**) Let us consider the following sequence defined recursively by

$$a_1 = \sqrt{3},$$
$$a_n = \sqrt{2a_{n-1} + 3}, \qquad n \geq 2.$$

We will prove that this sequence is convergent by showing that it is bounded and increasing.

(i) We prove by induction that
$$0 < a_n \leq 3,$$
for all $n \in \mathbb{N}$. This is true for $n = 1$. Assume that for $n = k$,
$$0 < a_k = \sqrt{2a_{k-1} + 3} \leq 3.$$
Hence,
$$0 < a_{k+1} = \sqrt{2a_k + 3} \leq \sqrt{6 + 3} = 3.$$
So, our sequence is bounded. **[5]**

(ii) We now prove that it is increasing, i.e., $a_{n+1} \geq a_n$, for all $n \in \mathbb{N}$. This is equivalent to prove that
$$\sqrt{2a_n + 3} \geq a_n \quad \Leftrightarrow 2a_n + 3 \geq a_n^2 \quad \Leftrightarrow \quad a_n^2 - 2a_n - 3 = (a_n - 3)(a_n + 1) \leq 0.$$
The last inequality holds because $0 < a_n \leq 3$. **[5]**

(iii) By a theorem seen in class on monotone sequences, we know the sequence (a_n) is convergent because it is bounded and increasing and
$$\lim_{n\to\infty} a_n = \sup_n a_n = x$$
By the shift rule,
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n-1} = x.$$
So, taking the limit at both sides of the equality
$$a_n^2 = 2a_{n-1} + 3,$$
we get
$$x^2 - 2x - 3 = 0.$$
This gives $x = 3$ or $x = -1$. Since x must be positive we have that $x = 3$. Concluding, $a_n \to 3$ as $n \to \infty$. **[5]**

Question 3 [25 marks].

(a) Decide whether the following series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, conditionally convergent or divergent. Carefully justify your answer.

(i)
$$a_n = \frac{\sin n}{(n+2)(n+3)} + \frac{2^n + 5^n}{2^n + 9^n};$$
[7]

(ii)
$$a_n = (-1)^n \frac{1}{\sqrt{\sqrt{n} + 4}}.$$
[7]

(b) (i) Find the radius of convergence R of the series
$$\sum_{n=0}^{\infty} (-2)^n n^4 (x-1)^n;$$
[5]

(ii) What can you say for the series (i) when $x = 1 \pm R$? Justify your answer. **[6]**

Question 3

(a) Decide whether the following series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, conditionally convergent or divergent. Carefully justify your answer.

(i)
$$a_n = \frac{\sin n}{(n+2)(n+3)} + \frac{2^n + 5^n}{2^n + 9^n};$$

[**7**]

Solution: (**Similar seen in class**) Since

$$a_n = \frac{\sin n}{(n+2)(n+3)} + \frac{2^n + 5^n}{2^n + 9^n};$$

is the sum of two terms we will argue on them separately. So we write

$$a_n = b_n + c_n, \qquad b_n = \frac{\sin n}{(n+2)(n+3)}, \quad c_n = \frac{2^n + 5^n}{2^n + 9^n}$$

It follows that
$$0 \leq |a_n| \leq |b_n| + |c_n| = |b_n| + c_n.$$

So, if we prove that the series given by $|b_n|$ and c_n are both convergent we will have that the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. To prove that $\sum_{n=1}^{\infty} |b_n|$ is convergent we use the comparison test. We have

$$0 \leq \left| \frac{\sin n}{(n+2)(n+3)} \right| \leq \frac{1}{(n+2)(n+3)}.$$

The telescopic series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent so by the shift rule $\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)}$. is convergent too. By the comparison test $\sum_{n=1}^{\infty} |b_n|$ is convergent as well. [**3**]

For the series $\sum_{n=1}^{\infty} c_n$ we use the ratio test. We have to compute

$$\lim_{n\to\infty} \frac{2^{n+1} + 5^{n+1}}{2^{n+1} + 9^{n+1}} \frac{2^n + 9^n}{2^n + 5^n}.$$

Since

$$\frac{2^{n+1} + 5^{n+1}}{2^n + 5^n} = \frac{2^{n+1}}{2^n + 5^n} + \frac{5^{n+1}}{2^n + 5^n} = \frac{2}{1 + \frac{5}{2}^n} + \frac{5}{1 + \frac{2}{5}^n} \to 5$$

and

$$\frac{2^n + 9^n}{2^{n+1} + 9^{n+1}} = \frac{2^n}{2^{n+1} + 9^{n+1}} + \frac{9^n}{2^{n+1} + 9^{n+1}} = \frac{\frac{1}{2}}{1 + \frac{9}{2}^{n+1}} + \frac{\frac{1}{9}}{1 + \frac{2}{9}^{n+1}} \to \frac{1}{9}$$

we have that
$$\lim_{n\to\infty} \frac{2^{n+1} + 5^{n+1}}{2^{n+1} + 9^{n+1}} \frac{2^n + 9^n}{2^n + 5^n} = \frac{5}{9} < 1$$

so, by the ratio test the series defined by c_n is convergent as well. This yields that the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. [**4**]

(ii)

$$a_n = (-1)^n \frac{1}{\sqrt{\sqrt{n}+4}}.$$

[**7**]

Solution (**Similar seen in class**) Let us consider the series defined by

$$a_n = (-1)^n \frac{1}{\sqrt{\sqrt{n}+4}}.$$

Since the sequence $\frac{1}{\sqrt{\sqrt{n}+4}}$ is positive, decreasing and convergent to 0 we can apply the alternating series test and conclude that

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{\sqrt{n}+4}}$$

is convergent. [**5**]

Let us now study the series of the absolute values, i.e.,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{\sqrt{n}+4}}$$

The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{\sqrt{n}+4}}$ behaves like the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{\sqrt{n}}}$ which is divergent. So, by the comparison test the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{\sqrt{n}+4}}$$

is divergent. This proves that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{\sqrt{n}+4}}$$

is conditionally convergent. [**6**]

(b) (i) Find the radius of convergence R of the series

$$\sum_{n=0}^{\infty} (-2)^n n^4 (x-1)^n;$$

[**5**]

Solution (**Similar seen in class**)

To determine the radius of convergence of the power series

$$\sum_{n=0}^{\infty} (-2)^n n^4 (x-1)^n;$$

we can use the root test. This means that we want to compute the limit

$$\lim_{n\to\infty} |(-2)^n n^4|^{\frac{1}{n}} = \lim_{n\to\infty} 2n^{\frac{4}{n}}.$$

We have

$$\lim_{n\to\infty} n^{\frac{4}{n}} = \lim_{n\to\infty} \mathrm{e}^{\ln(n^{\frac{4}{n}})} = \lim_{n\to\infty} \mathrm{e}^{\frac{4\ln n}{n}} = 1.$$

So

$$\lim_{n\to\infty} |(-2)^n n^4|^{\frac{1}{n}} = 2$$

and the radius of convergence is $R = \frac{1}{2}$.

(ii) What can you say for the series (i) when $x = 1 \pm R$? Justify your answer. [**6**]

Solution (**Similar seen in class**)

In our case we have $x = 1 \pm \frac{1}{2}$, i.e., $x = \frac{1}{2}$ and $x = \frac{3}{2}$.

With $x = \frac{1}{2}$ we get the series

$$\sum_{n=0}^{\infty} (-2)^n n^4 \frac{1}{(-2)^n} = \sum_{n=0}^{\infty} n^4$$

is divergent because $n^4 \to \infty$. [**3**]

For $x = \frac{3}{2}$ the series

$$\sum_{n=0}^{\infty} (-2)^n n^4 \frac{1}{2^n} = \sum_{n=0}^{\infty} (-1)^n n^4$$

is divergent because $(-1)^n n^4 \not\to 0$. [**3**]

Question 4 [25 marks].

(a) Prove that the equation

$$x + \ln(\sin(x) + 2) = 1$$

has a solution $x \in \mathbb{R}$. [**15**]

(b) Can you find a continuous function $f : [0, 1] \to \mathbb{R}$ such that for all $c > 0$ there exists $x \in [0, 1]$ such that

$$|f(x)| > c?$$

Justify your answer. [**10**]

Question 4

(a) Prove that the equation

$$x + \ln(\sin(x) + 2) = 1$$

has a solution $x \in \mathbb{R}$. **[5]**

Solution (Similar seen in class)

We want to apply the Intermediate Value Theorem to the function

$$g(x) = x + \ln(\sin(x) + 2) - 1$$

This is a continuous function on $\mathbb{R}$ by composition and sum of continuous functions. I want to find an interval $[a, b]$ such that $g(a) < 0$ and $g(b) > 0$. Then, by applying the theorem we will have a point $c \in (a, b)$ such that $g(c) = 0$. This will be the solution to our equation. For instance, we can choose $a = -\frac{\pi}{2}$ and $b = \frac{\pi}{2}$. Hence,

$$g(-\frac{\pi}{2}) = -\frac{\pi}{2} - 1 < 0$$

and

$$g(\frac{\pi}{2}) = \frac{\pi}{2} + \ln(3) - 1 > 0.$$

[15]

(b) Can you find a continuous function $f : [0, 1] \to \mathbb{R}$ such that for all $c > 0$ there exists $x \in [0, 1]$ such that

$$|f(x)| > c?$$

Justify your answer. **[10]**

Solution (Unseen)

The answer is no. Since the function f is continuous on $[0, 1]$ it is also bounded. So, there exists $c > 0$ such that

$$|f(x)| \leq c$$

for all $x \in [0, 1]$. **[10]**

End of Paper.