Queen Mary University of London

Main Examination period 2024 – May/June – Semester B

MTH5104: Convergence and Continuity (Practice Exam)

Examiners: N. Nabijou

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You will have a period of **3 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

The exam is closed-book, and **no outside notes are allowed**.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any unauthorised notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately.

It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it will be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

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Important: All your answers must be justified. Unless the question explicitly indicates otherwise, you may use any result from the lectures, provided you state the result clearly.

Question 1 [25 marks].

(a) Given a subset $A \subseteq \mathbb{R}$ and a real number $a \in \mathbb{R}$, define what it means for a to be the supremum of A. [**5**]

(b) Consider the following set of real numbers:

$$A = \left\{ \frac{3n^2 + n + 3}{n^3 + n} : n \in \mathbb{N} \right\} \subseteq \mathbb{R}.$$

(i) Prove that A is bounded. [**4**]

(ii) Prove that $\inf(A) = 0$. Can you replace inf by min? Justify your answer. [**4**]

(iii) Find $\sup(A)$ and $\max(A)$, justifying your answer in each case. [**3**]

(b) Let $B \subseteq \mathbb{R}$ be nonempty and bounded.

(i) Prove that $\inf(B) \leq \sup(B)$. [**5**]

(ii) Does there exist a B such that $\inf(B) = \sup(B)$? [**4**]

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Question 2 [25 marks].

(a) Define what it means for a sequence (x_n) to converge to a value $x \in \mathbb{R}$. [**5**]

(b) Let (x_n) and (y_n) be sequences, and suppose that $x_n \to \infty$ and $y_n \to \infty$. Prove that $x_n + y_n \to \infty$. [**6**]

(c) Give an example of a sequence that contains both a bounded subsequence and an unbounded subsequence. [**5**]

(d) For each of the following sequences, decide whether or not it converges, and justify your answer.
(You may use any result from the lectures, but you must state the result clearly.)

(i) $x_n = \dfrac{1}{n^{2023}}$. [**3**]

(ii) $x_n = \dfrac{1}{n + n^2}$. [**3**]

(iii) $x_n = n(2 + \sin(n\sqrt{2023}))$. [**3**]

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Question 3 [25 marks].

(a) Define what it means for a series to be conditionally convergent. [**6**]

(b) For each of the following series, decide whether or not it converges. You do not need to calculate its value.

(i) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k + 3^k}{2^k + 7^k}$. [**6**]

(ii) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2 + \sqrt{k} + 1}$. [**6**]

(c) Suppose we are given two conditionally convergent series $\sum_{k=1}^{\infty} x_k$ and $\sum_{k=1}^{\infty} y_k$. Does it follow that the series

$$\sum_{k=1}^{\infty} (x_k + y_k)$$

is conditionally convergent? Prove or give a counterexample. [**7**]

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Question 4 [25 marks].

(a) Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by:

$$f(x) = 3x + 4.$$

Prove, **directly from the definition**, that $f(x)$ is continuous at all points $a \in \mathbb{R}$. [**6**]

(b) Consider the function $g\colon \mathbb{R} \to \mathbb{R}$ given by:

$$g(x) = \begin{cases} \sqrt{x} & \text{if } x > 0 \\ -1 & \text{if } x \leq 0 \end{cases}$$

Find a point $a \in \mathbb{R}$ such that $g(x)$ is not continuous at a. Justify your answer. (You may use any result from the lectures, but you must state the result clearly.) [**6**]

(c) Consider now two arbitrary functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Suppose that $f(x)$ and $g(x)$ are both continuous at the point $a \in \mathbb{R}$. Prove **directly from the definition** that the function

$$h(x) = f(x) + g(x)$$

is also continuous at a. [**7**]

(d) Prove that there exists an $x \in \mathbb{R}$ such that $x + \cos(x) = 1$. [**6**]

End of Paper.

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