Queen Mary
University of London

MTH5104: Convergence and Continuity 2023–2024
Problem Sheet 6 (Continuity)

1. Prove, directly for the definition of continuity, that $f(x) = \sqrt[3]{x}$ is continuous at $a = 0$.

2. For each of the following functions, state whether they are continuous at $a = 0$ and *prove* your answers, using only the definition of continuity.

   (a) $f(x) = \begin{cases} 2x & \text{if } x \in \mathbb{Q}, \\ -5x & \text{if } x \notin \mathbb{Q}, \end{cases}$

   (b) $f(x) = \begin{cases} 2x+1 & \text{if } x \geq 0, \\ -5x & \text{if } x < 0. \end{cases}$

3. The function $f : \mathbb{R} \to \mathbb{R}$ is defined by:

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Z}, \\ x & \text{if } x \in \mathbb{Z}. \end{cases}$$

   Find the set $P \subseteq \mathbb{R}$ of points for which the function f is not continuous. Prove your answer!

4. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 + x$. Prove, directly from the definition, that $f(x)$ is continuous at all $a \in \mathbb{R}$. (There is an example in the notes that can be used as a model. Given ε, you may like to try letting $\delta = \min\{c\varepsilon, 1\}$ for some suitably chosen constant $c \in \mathbb{R}$.)

   (b) Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} x^2 + x, & \text{if } x \text{ is rational} \\ 0, & \text{otherwise.} \end{cases}$$

   The function g is continuous at two points; what are they?

   (c) For one of the points a you identified in part (b), verify that g is continuous at a. (This part requires very little calculation. There are two cases, $a + h \in \mathbb{Q}$ and $a + h \notin \mathbb{Q}$. In part (a) you already did the work for the harder one of these!)

5. Prove parts (i) and (ii) of Theorem 5.14 from the lecture notes.

6. For each of the following real functions state its natural domain of definition D. Also determine at which points of D the function is continuous. Explain your answers by reference to results in the course. You may assume that $\ln x$ is defined and continuous at all points in $(0, \infty)$.

   (a) $f(x) = \cos\left(\dfrac{1}{x^2+1}\right)$,

   (b) $g(x) = \ln(\ln x)$, and

   (c) $h(x) = \sqrt{\dfrac{x}{x^2+1}}$.

7. Consider the function $f(x) = \sqrt{x}$ defined on $D = [0, \infty)$. Prove, directly from the definition of continuity, that $f(x)$ is continuous on D. (First, try showing that $|\sqrt{x} - \sqrt{a}\,|^2 \leq |x - a|$.)

8. **Challenge.** Prove parts (iii) and (iv) of Theorem 5.14 from the lecture notes.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \text{ or if } x = 0, \\ 1/q & \text{if } x \in \mathbb{Q} \text{ and } x = p/q \text{ in lowest terms, with } p > 0. \end{cases}$$

   (a) Suppose that $a \in \mathbb{Q}$. Prove that f is not continuous at a.

   (b) Suppose that $a \notin \mathbb{Q}$. Prove that f is continuous at a [harder].

10. Using the intermediate value theorem, show that the following equations have a solution $x \in \mathbb{R}$:

    (a) $x^5 + 2x^2 = 1$.

    (b) $x^4 + 1 = 9x$.

    (c) $x\cos(x) + x^2 = 1$.

11. Find a continuous map of the open interval $(0, 1) \subset \mathbb{R}$ to itself which has no fixed point in $(0, 1)$. This shows that the analogue of the Brouwer Fixed Point Theorem for *open* intervals is not true.

12. Let $f : [0,1] \to \mathbb{R}$ be a continuous function which has the property that $f(0) = f(1)$. Let $g : [0,1/2] \to \mathbb{R}$ be defined by $g(x) = f(x+1/2) - f(x)$. Show that $g(0) + g(1/2) = 0$. By applying the Intermediate Value Theorem to g, prove that there exists a real number $c \in [0,1/2]$ such that $f(c+1/2) = f(c)$.

13. Suppose that $f : [0,1] \to \mathbb{R}$ and that $(x_n)_{n=1}^{\infty}$ is the sequence of all rationals ordered by increasing denominator

$$0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \ldots$$

and that $f(x_n) = n$. Why do we know, *without doing any calculation* that f is not continuous? (This question does not ask for a precise proof, just an explanation.)

14. In this question, be sure to check that all the conditions of the Intermediate Value Theorem hold.

    (a) Prove that, for every $c \in [0,\infty)$, the equation $xe^x = c$ has a solution.

    (b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function that is bounded, i.e, $|f(x)| \leq M$ for all $x \in \mathbb{R}$. Prove that the function $f(x)$ has a fixed point in $\mathbb{R}$.

15. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} x - 1 & \text{if } x \in \mathbb{Q}, \\ x + 1 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

    (a) What does the sequence $(x_n)_{n=1}^{\infty}$ given by $x_n = \frac{n-\sqrt{2}}{n}$ converge to?

    (b) Does the sequence $(f(x_n))_{n=1}^{\infty}$ converge? If so what does it converge to?

    (c) Is f continuous at the point $a = 1$? (Give a brief justification.)

16. **Challenge**. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Recall from Definition 5.22 that we say that $\lim_{x\to a} f(x)$ exists and is equal to ℓ iff

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall h \in \mathbb{R}, 0 < |h| < \delta\ :\ |f(a+h) - \ell| < \varepsilon. \qquad (\star)$$

    (a) Show that $\lim_{x\to a} f(x) = \ell$ (according to the above definition) if and only if for *every* sequence $(x_n)_{n=1}^{\infty}$ which satisfies $x_n \neq a$ for all n as well as $x_n \to a$ for $n \to \infty$, we get $f(x_n) \to \ell$ as $n \to \infty$.

    (b) Show that f is continuous at a (according to our Definition 5.1) if and only if $\lim_{x\to a} f(x)$ exists and is equal to $f(a)$.