Random Processes – 2023/24 Problem Sheet 10

The questions on this sheet are based on the material on continuous-time Markov chains from Week 11 and 12 lectures. This will be the final problem sheet for the module and it is slightly longer than usual as it covers two weeks material.
To help navigate the sheet: Questions 2 and 3 are about the generator matrix; Questions 4 and 5 extend some of the theory of queues from lectures; Question 6 develops some of the theory of long-term behaviour for continuous-time chains with finite state spaces[1]; finally Questions 7 and 8 are about modifications to processes we have seen so far, these two questions are a little niche and you could leave them out without missing much
If you are struggling with this material then I would recommend initially focusing on Questions 2, 3 and 6.
I would like you to submit your answer to **Question 1** by **5pm on Friday 15 December 2023** via the QMplus page following the instructions there.

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1. Which concept from this module are you most curious to know more about? Give a (general or specific) mathematical question which you would like to know the answer to.

2. Let X_t be the discrete-time Markov chain on state space $\{1, 2, 3, 4\}$ with transition matrix

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/4 & 1/2 & 0 & 1/4 \\ 1/2 & 1/3 & 1/6 & 0 \end{pmatrix}$$

Define a continuous-time process $(Y(t) : t \geqslant 0)$ on the same state space as follows. When this process enters a state it waits for a random time with distribution $\text{Exp}(2)$ and then performs one step of the chain X_t.

(a) Write down the generator matrix for $Y(t)$.

Define a continuous-time process $(Z(t) : t \geqslant 0)$ on the same state space as follows. When this process enters state k it waits for a random time with distribution $\text{Exp}(k)$ and then performs one step of the chain X_t.

(b) Write down the generator matrix for $Z(t)$.

[1] This would probably be the next thing I would lecture if we had more time and it's such a nice thing I can't resist giving you a taste of it on the problem sheet

3. Suppose that all the emails I receive have 0, 1, 2 or 3 attachments. Emails with k attachments arrive as a Poisson process of rate α_k. Let $A(t)$ be the number of attachments in all emails I receive in the time interval $[0, t]$.

(a) Derive the transition rates g_{ij} $(i \neq j)$ for this process.

(b) Write down the (infinite) generator G for this process.

4. In lectures (Examples 50 and 54) we showed that the $M(\lambda)/M(\mu)/1$ is a birth-death process and derived a limiting distribution. This question leads you through a similar analysis for the $M(\lambda)/M(\mu)/2$ queue. This is a queue with arrivals and service times exactly as for $M(\lambda)/M(\mu)/1$ but now there are 2 servers. When a customer reaches the front of the queue they wait until one of the servers becomes available and then moves to that server.
Let $Q(t)$ be the total number of people in the system (waiting and being served) at time t

(a) Show that this is a birth-death process with parameters

$$\lambda_i = \lambda, \quad \mu_i = \begin{cases} 0 & \text{if } i = 0; \\ \mu & \text{if } i = 1; \\ 2\mu & \text{if } i \geqslant 2. \end{cases}$$

(b) Write down the forwards and backwards equations for $Q(t)$.

(c) Assuming that there is a limiting distribution

$$\mathbb{P}(Q(t) = j \mid Q(0) = i) \to w_j \text{ as } t \to \infty,$$

use part (b) to find equations for the w_j.

(d) Show that these have solution $w_j = 2\rho^j w_0$ for $j \geqslant 1$ where $\rho = \frac{\lambda}{2\mu}$.

(e) Deduce that there is a limiting distribution when $\lambda < 2\mu$.

(f) Finally, determine the w_j in the case that $\lambda < 2\mu$.

5. Suppose that the $M(\lambda)/M(\mu)/1$ model is modified so that if there are n customers in the system when a new customer arrives, they join the queue as usual with probability p_n and otherwise leave in disgust.

(a) Say why this does not satisfy the assumptions we made in lectures.

(b) Nevertheless, show that this queue is described by a birth-death process. What are the parameters?

(c) How could you model a queue with a waiting room which can only hold k people (with anyone finding the room full leaving in disgust) using this approach?

6. This result takes you through some of the theory of long-term behaviour for continuous-time Markov chains on finite state spaces (analogous to parts of section 4.4–4.6)

(a) Write down an example of a generator matrix for a continuous-time Markov chain on state space $S = \{1, 2, 3\}$ in which every entry of the matrix is non-zero.

Now assume that the process with this generator has a limiting distribution. That is for all $i, j \in S$:

$$\mathbb{P}(X(t) = j \mid X(0) = i) \to w_j \text{ as } t \to \infty$$

(note the limit does not depend on i).

(b) Under this assumption use the backwards equations to show that

$$\lim_{t\to\infty} p'_{i,j}(t) = 0$$

for all $i, j \in S$.

(c) Now use part (b) and the forwards equations to get a matrix equation for $\begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix}$ and solve it to determine w_1, w_2 and w_3.

(d) Finally, think about what would happen if we replaced your 3-state example with any continuous-time Markov chain with $S = \{1, 2, \ldots, n\}$. Write down what the general version of the argument you have given proves. What result from the theory of discrete-time Markov chains is this similar to?

7. Let $(X(t) : t \geqslant 0)$ be a Poisson process of rate λ. Let $(Y(t) : t \geqslant 0)$ be the continuous-time process on state space $\{0, 1\}$ defined by

$$Y(t) = \begin{cases} 0 & \text{if } X(t) \text{ is even;} \\ 1 & \text{if } X(t) \text{ is odd.} \end{cases}$$

(a) Describe the process $Y(t)$.

(b) What happens if we make a similar construction where $X(t)$ is a birth process?

8. An island is initially uninhabited. At the end of every day a boat arrives on the island (so the ith boat arrives at time i). Each boat brings one creature to the island with probability p. Once there each creature produces offspring as a Poisson process of rate β per day independently of the arrival of boats and all other births. Assume also that there are no deaths and no creatures leave the island. Let $X(t)$ be the number of creatures on the island at time t days.

(a) Say what is wrong with the following argument:

> The arrival rate is p per day. So just as in Question 1 on Sheet 9, this is a birth process with parameters $\lambda_k = p + \beta k$.

(b) Suppose that $p = 1$. What is the distribution of the time of the first arrival?

(c) Suppose that $p < 1$. What is the distribution of the time of the first arrival?

(d) Suppose that $p = 1$. What is the distribution of the time the population size first reaches 2?

Here are some recent exam questions on the material in Week 11 and 12. As indicated some parts of these paper are on material not covered this year.

- January 2020 Exam. Question 5 (Part (e) not covered this year)
- January 2021 Exam. Question 4 (Part (d) not covered this year)
- January 2022 Exam. Question 3(f)
- January 2023 Exam. Question 4(b,c)

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