Random Processes – 2023/24 **Problem Sheet 9**

The questions on this sheet are based on the material on birth processes from Week 10 lectures.

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1. An island is initially uninhabited. Creatures arrive as a Poisson process of rate α per month. Once there each creature independently produces offspring as a Poisson process of rate β per month. Assume that the Poisson processes for arrivals and births are all independent, that there are no deaths and no creatures leave the island. Let $X(t)$ be the number of creatures on the island at time t months and

$$p_n(t) = \mathbb{P}(X(t) = n).$$

(a) What is the probability that the island is inhabited at time 3 months?

(b) Explain why

$$\mathbb{P}(X(t+h) = m \mid X(t) = m) = (1 - \alpha h)(1 - \beta h)^m + o(h)$$

and simplify the expression on the righthand side.

(c) Similarly, find an expression for $\mathbb{P}(X(t+h) = m+1 \mid X(t) = m)$.

(d) Indicate briefly why $\mathbb{P}(X(t+h) \geqslant m+2 \mid X(t) = m) = o(h)$.

(e) Explain why $X(t)$ is a birth process. What are the birth parameters?

2. Let $(X(t) : \ t \geqslant 0)$ be the size of a population given by a birth process with $X(0) = 0$ and birth parameters $\lambda_i = 3 + i$.

(a) Write down the differential equations for $p_0'(t)$, $p_1'(t)$ and $p_2'(t)$ from lectures (you can quote the result from lectures without repeating the derivation).

(b) Solve the three equations in part (a).

(c) What is the probability that the population has size at least 3 at time 1?

(d) What is the expectation of the time when the population size first reaches 10? [Hint: Mercifully you don't need to determine $p_{10}(t)$ to do this!]

(e) Explain how the birth parameters $\lambda_i = 3 + i$ could arise in practice.

3. Let $X(t)$ be a linear birth process (that is $\lambda_n = n\lambda$ for $n \geqslant 1$) and $X(0) = 1$. Let $p_n(t) = \mathbb{P}(X(t) = n)$ for $n \geqslant 1$.

(a) Write down the differential equations for the $p_n(t)$.

(b) Show that they have solution $p_n(t) = e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{n-1}$.

(c) Use this to write down the distribution of $X(t)$ and find its expectation.

(d) How does your answer to (c) compare with the corresponding expectation for a Poisson process as time t increases?

4. Let $Y(t)$ be a Poisson process of rate 2. Let $Z(t)$ be a a birth process with birth parameters $\lambda_i = 2^i$ and $Z(t) = 0$. Suppose that $Y(t)$ and $Z(t)$ are independent processes.

(a) For each k determine whether the expected time of the kth arrival in $Y(t)$ is smaller or larger that the expected time of the kth birth in $Z(t)$.

(b) What can you say about the probability of explosion in the two processes?

Birth processes haven't featured much on the exam but some recent exam questions on the material in Week 9 include:

- January 2022 Exam. Question 3 (a-d), Question 4(b)
- January 2023 Exam. Question 4(c) (possibly)

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