Random Processes – 2023/24 Problem Sheet 8

The questions on this sheet are based on the material on the Poisson process from Week 8 and 9 lectures.

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1. A 90 minute football match consisting of two 45 minutes halves takes place between two teams A and B. During the match, Team A makes shots at goal according to a Poisson process of rate 1/10 per minute and Team B makes shots at goal according to a Poisson process of rate 1/12 per minute. These Poisson processes are independent. Independently of all other events, each of Team A's shots results in a goal with probability 1/3, and each of Team B's shots results in a goal with probability 1/2. We will write these processes as:

- $X_A(t)$ =number of shots at goal for team A in time $[0, t]$
- $X_B(t)$ =number of shots at goal for team B in time $[0, t]$
- $Y_A(t)$ =number of goals for team A in time $[0, t]$
- $Y_B(t)$ =number of goals for team B in time $[0, t]$
- $G(t)$ =total number of goals in time $[0, t]$

(a) What can you say about the process $(G(t) : \ t \geqslant 0)$? Say which results from lectures you use in each step.

(b) What is the probability that Team A wins the match by 3 goals to 2?

(c) I arrive at the start of the match. What is the expectation of the time I wait before I see team A have a shot at goal?

(d) My friend arrives 10 minutes after the match has started. What is the expectation of the time she waits until she sees team A have a shot at goal?

(e) Suppose that there are exactly 4 goals in the match. What is the probability that more that half of them are scored in the first half?

(f) Translate the following information into a non-mathematical description of how the match developed:

- $X_A(1) = Y_A(1) = 1$
- $X_A(45) = 1$, $X_B(45) = 0$
- $\min\{t : Y_B(t) = 1\} = 72$
- $X_A(90) - X_A(89) = 2$, $Y_A(90) - Y_A(89) = 0$
- $Y_A(90) = Y_B(90) = 1$

2. Let $X(t)$ be a Poisson process of rate λ.

(a) What is $\mathbb{P}(T_1 \leqslant u)$?

(b) Suppose that $n \geqslant 1$ and $0 \leqslant u \leqslant t$. Find $\mathbb{P}(T_1 \leqslant u \mid X(t) = n)$?

(c) Comment on how changing λ changes the answers to parts (a) and (b).

(d) Show that for all $n \geqslant 1$ the probability density function of T_1 conditioned on $X(t) = n$ is

$$f_{T_1|X(t)=n}(u) = \frac{n}{t}\left(1 - \frac{u}{t}\right)^{n-1} \quad \text{for } 0 < u \leqslant t.$$

3. Let $T_1, T_2, \ldots$ be the arrival times of a Poisson process $X(t)$ of rate λ. Find the following:

(a) $\mathbb{E}(T_1 + T_2 + T_3 \mid X(10) = 3)$,

(b) $\mathbb{E}(T_1^2 T_2^2 T_3^2 T_4^2 \mid X(1) = 4)$,

4. Requests arrive at a server as a Poisson process of rate λ per minute. Every T minutes the requests are processed regardless of how many there are (even if there are none). Suppose that processing costs £k (regardless of how many requests are processed). In addition each request incurs a cost of £c for each minute it waits before processing.

(a) Show that the expected cost per minute to run the server is $\frac{k}{T} + \frac{c\lambda T}{2}$.

[Hint: First condition on there being n requests waiting at time T.]

(b) How should T be chosen to minimize the cost of running the server?

5. [Challenge Question] I arrive at a bus stop at a random time. What is the expected time I must wait for my bus under the following assumptions:

(a) I am in London, and the buses arrive according to a Poisson process with rate 6 per hour?

(b) I am in Zürich, and the buses are equally spaced in time, 10 minutes apart?

Notice that in each case the expected number of buses per hour is the same (it is 6 in each case).
How would you explain the difference in your answers to the two parts to a non-mathematician?

Some recent exam questions on the material in Week 9 include:

- Main Exam Period 2018. Question 6 (e,f)
- Main Exam Period 2019. Question 4(f)
- January 2020 Exam. Question 4
- January 2022 Exam. Question 3 (a-d), Question 4(b)
- January 2023 Exam. Question 2(c)

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