2.

(a) The key point here is that for any s, t we have that $C(s+t) - C(s) \sim \text{Po}(\frac{t}{2})$.

(i) $\mathbb{P}(C(10) = 3) = \mathbb{P}(C(10) - C(0) = 3) = e^{-5}\frac{5^3}{3!} = \frac{125}{6e^5}$

(ii)

$$\begin{aligned}
\mathbb{P}(C(10) = 3 \mid C(5) = 0) &= \mathbb{P}(C(10) - C(5) = 3 \mid C(5) - C(0) = 0) \\
&= \mathbb{P}(C(10) - C(5) = 3) \\
&\qquad \text{(by property (ii) of the Poisson process)} \\
&= e^{-5/2}\frac{(5/2)^3}{3!} \\
&= \frac{125}{48e^{5/2}}.
\end{aligned}$$

(iii)

$$\begin{aligned}
\mathbb{P}(C(10) = 3, C(5) = 0) &= \mathbb{P}(C(10) - C(5) = 3, C(5) - C(0) = 0) \\
&= \mathbb{P}(C(10) - C(5) = 3)\mathbb{P}(C(5) - C(0) = 0) \\
&= e^{-5/2}\frac{(5/2)^3}{3!}e^{-5/2}\frac{(5/2)^0}{0!} \\
&= \frac{125}{48e^5}.
\end{aligned}$$

(iv) In a Poisson process we must have $C(s) \geqslant C(t)$ for all $s \geqslant t$. So

$$\mathbb{P}(C(10) = 0 \mid C(5) = 3) = 0.$$

(b) By the thinning Lemma (Lemma 7.3), customers who make a purchase arrive according to a Poisson process of rate 1/4 per minute. Let

$B(t) =$ number of customers who make a purchase who arrive in the time interval $[0, t]$

Then $B(s+t) - B(s) \sim \text{Po}(\frac{t}{4})$.

(i) This asks for

$$\mathbb{P}(B(10) = 4) = e^{-10/4}\frac{(10/4)^4}{4!} = e^{-5/2}\frac{625}{384}$$

(ii) Each of the first 3 customers has probability $1/2$ of making a purchase and for each one of them whether they make a purchase or not is independent of the other customers. So this probability is $(1/2)^3 = 1/8$.

(iii) This asks for $\mathbb{E}(B(600))$. We know that $B(600) \sim \text{Po}(600/4)$. So

$$\mathbb{E}(B(600)) = 600/4 = 150.$$

(c) If each customer spends 5 minutes in the shop then the customers present at time 1 hour are those who entered the shop in the interval $[55, 60]$. The number of these has $\text{Po}(5/2)$ distribution.

3.

(a) Let $H(t)$ be the number of Hammersmith and City Line trains arriving up to time t and $D(t)$ be the number of District Line trains arriving in time up to t. The total number of trains arriving up to time t is $U(t) = H(t) + D(t)$. Let $F(t)$ be the total number of full trains arriving up to time t.

The questions tells us that $H(t)$ is a Poisson process of rate 10 per hour, $D(t)$ is a Poisson process of rate 15 per hour, and these processes are independent. By Lemma 7.2 (superposition) we have that $U(t)$ is a Poisson process of rate $10 + 15 = 25$.

Let $F(t)$ be the number of full trains arriving up to time t. By Lemma 7.3 (thinning) we have that $F(t)$ is a Poisson process of rate $25/10 = 2.5$.

(b) Let $X(t)$ be the number of Hammersmith and City line trains which are not full which arrive up to time t. Similarly to part (a), we have $X(t)$ is a Poisson process of rate $10 \times 9/10 = 9$. To say that after 6 minutes, no Hammersmith and City line train which is not full has arrived means that $X(1/10) = 0$. Now

$$\mathbb{P}(X(1/10) = 0) = e^{-\frac{9}{10}} = 0.41 \quad \text{to 2 decimal places}$$

So waiting this long happens around $2/5$ of the time.

4.

(a) We have

$$\mathbb{P}(\text{no arrivals in } [0,1]) = e^{-\lambda}$$

and

$$\mathbb{P}(\text{one arrival in } [0,1]) = e^{-\lambda}\lambda.$$

So if these are equal we have

$$e^{-\lambda} = e^{-\lambda}\lambda$$

which means $\lambda = 1$.

(b) Similarly,

$$\mathbb{P}(\text{two arrivals in } [0,1]) = e^{-\lambda}\frac{\lambda^2}{2}$$

so if $\mathbb{P}(\text{no arrivals in } [0,1]) = \mathbb{P}(\text{two arrivals in } [0,1])$ we have

$$e^{-\lambda} = e^{-\lambda}\frac{\lambda^2}{2}$$

which means $\lambda = \sqrt{2}$.

(c)

$$\mathbb{P}(\text{no arrivals in } [0,3]) = e^{-3\lambda}$$

so if $\mathbb{P}(\text{no arrivals in } [0,1]) = 2\mathbb{P}(\text{no arrivals in } [0,3])$ we have

$$e^{-\lambda} = 2e^{-3\lambda}.$$

Solving this (take logs and rearrange) we get $\lambda = \frac{\log 2}{2}$ (where log indicates natural logarithm).

(d)

$$\mathbb{P}(\text{no arrivals in } [1,2]) = e^{-\lambda} = \mathbb{P}(\text{no arrivals in } [0,1])$$

So this equality holds for any λ. The condition gives us no information about the rate.

5.

(a) Let's take midnight to be time 0 and let

$$X(t) = \text{the number of emails received up to time } t$$

We will split $\mathbb{R}$ into $D = \{t \in \mathbb{R} : 8k \leqslant t \leqslant 18k, \text{ for some } k \in \mathbb{N}\}$ (times during the day 8am to 6pm) and $N = \mathbb{R} \setminus D$ (times during the night 6pm to 8am).

For a small interval of length h, the chance of receiving an email should be $3h$ if the interval is in D and $0.2h$ if the interval is in N. So we should change the infinitesimal definition to:

- for $t \in D$ let

$$\mathbb{P}(X(t+h) = n \mid X(t) = m) = \begin{cases} 3h + o(h) & \text{if } n = m+1; \\ 1 - 3h + o(h) & \text{if } n = m; \\ o(h) & \text{if } n \geqslant m+2; \\ 0 & \text{if } n < m. \end{cases}$$

- and for $t \in N$ let

$$\mathbb{P}(X(t+h) = n \mid X(t) = m) = \begin{cases} 0.2h + o(h) & \text{if } n = m+1; \\ 1 - 0.2h + o(h) & \text{if } n = m; \\ o(h) & \text{if } n \geqslant m+2; \\ 0 & \text{if } n < m. \end{cases}$$

(b)
- If the interval $[s, s+t]$ is contained completely within D then the number of emails received during it has distribution $\mathrm{Po}(3t)$.
- If the interval $[s, s+t]$ is contained completely within D then the number of emails received during it has distribution $\mathrm{Po}(0.2t)$.
- If the interval $[s, s+t]$ has a of its length in D and b of its length in N then the number of emails received during it is the sum of a $\mathrm{Po}(3a)$ randomv variable and an independent $\mathrm{Po}(0.2b)$ random variable. So it has distribution $\mathrm{Po}(3a + 0.2b)$.

Please let me know if you have any comments or corrections

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