Random Processes – 2023/24 **Problem Sheet 7**

The questions on this sheet are based on the material on the Poisson process from Week 8 lectures. We will discuss selected parts in the Week 9 seminars.

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1. Give an example of a real-world process (different from examples in lectures) for which you would expect a Poisson process to be a plausible approximation. Why would you expect reality to only approximately (rather than exactly) follow your model? Illustrate the concept of thinning using your example.

2. Customers enter a shop according to a Poisson process of rate 1/2 per minute. Let $C(t)$ be the number of customers who have entered the shop after it has been open for t minutes.

(a) Calculate the following:

  (i) $\mathbb{P}(C(10) = 3)$
  (ii) $\mathbb{P}(C(10) = 3 \mid C(5) = 0)$
  (iii) $\mathbb{P}(C(10) = 3, C(5) = 0)$
  (iv) $\mathbb{P}(C(10) = 0 \mid C(5) = 3)$

  (Leave any powers of e in your answer but simplify it in all other ways.)

(b) Suppose that each customer makes a purchase with probability 1/2 independently of the time they entered and whether each other customer made a purchase. Calculate the following:

  (i) The probability that the shop makes 4 sales from the customers who enter the shop in the first 10 minutes.
  (ii) The probability that the first 3 customers all make a purchase.
  (iii) The expectation of the number of sales the shop makes in a day during which it is open for 10 hours.

(c) Suppose that each customer spends exactly 5 minutes in the shop. Find the distribution of the number of customers in the shop after it has been open for 1 hour.

3. At Stepney Green Underground station the arrival of Hammersmith and City Line trains forms a Poisson process of rate 10 per hour and the arrival of District Line trains forms a Poisson process of rate 15 per hour. These processes are independent. Suppose that each train is full with probability $1/10$ independently of all other trains.

(a) What can you say about arrivals of full trains at Stepney Green Underground station. (Say which results from lectures you are using).

(b) I have been waiting at Stepney Green Underground station for 6 minutes, and up to that point no Hammersmith and City train with space has arrived. How likely is this to happen?

4. Suppose that $X(t)$ is a Poisson process. For each of the following conditions, decide what you can say about the rate of $X(t)$?

(a) $\mathbb{P}(\text{no arrivals in } [0,1]) = \mathbb{P}(\text{one arrival in } [0,1])$

(b) $\mathbb{P}(\text{no arrivals in } [0,1]) = \mathbb{P}(\text{two arrivals in } [0,1])$

(c) $\mathbb{P}(\text{no arrivals in } [0,1]) = 2\mathbb{P}(\text{no arrivals in } [0,3])$

(d) $\mathbb{P}(\text{no arrivals in } [0,1]) = \mathbb{P}(\text{no arrivals in } [1,2])$

5. [Challenge Question] I want to model the number of emails I receive during the day, taking into account the fact that more emails are sent during the day than at night. Suppose that between 8am and 6pm I receive emails randomly at a constant rate of 3 per hour, and between 6pm and 8am I receive emails randomly at a constant rate of 0.2 per hour.

(a) How could you modify the infinitesimal definition of the Poisson process to model this?

(b) What can you say about the distribution of the number of emails received in a particular interval of time?

Some recent exam questions on the material in Week 8 include:

- Main Exam Period 2018. Question 6(a-d)
- Main Exam Period 2019. Question 4
- January 2020 Exam. Question 4
- January 2021 Exam. Question 3
- January 2022 Exam. Question 3(a-d), Question 4(b)
- January 2023 Exam. Question 2(a,b)

Robert Johnson
r.johnson@qmul.ac.uk