Queen Mary
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MTH5104: Convergence and Continuity 2023–2024
Problem Sheet 4 (Sequences 2)

1. Which of the following sequences $(x_n)_{n=1}^{\infty}$ are monotonic? For those that are, state whether they are increasing or decreasing, and whether they are *strictly* increasing or decreasing. Give a brief justification in each case.

   (a) $x_n = \frac{n}{(n+1)(n+2)}$;

   (b) $x_n = \frac{n+1}{(n+2)(n+3)}$;

   (c) $x_n = \cos \pi n$;

   (d) $x_n = \lceil n^{1/2} \rceil$;

2. For each of the following three sequences state whether they converge to a limit. If a sequence converges, state and prove the limit. (You may use any results from the lecture notes but you should state which result you are using.) If the sequence does not converge, find two subsequences that converge to different limits. Again, justify your answer by reference to results from the lecture notes.

   (a) $(x_n)_{n=1}^{\infty}$, where $x_n = \big(1 + \sin(n\pi/5)\big)^2 \left(\dfrac{n+1}{2n^2}\right)$,

   (b) $(y_n)_{n=1}^{\infty}$, where $y_n = \dfrac{2n^2 + 5(-1)^n}{4n^2+1}$,

   (c) $(z_n)_{n=1}^{\infty}$, where $z_n = (-1)^n \left(\dfrac{n+3}{n+2}\right)$.

3. Prove the following "Sandwich principle" mentioned in the notes:

   If (y_n) is *some sequence* and (x_n) and (z_n) are two other sequences with $x_n \le y_n \le z_n$ for all $n \in \mathbb{N}$, and with $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = L$, then (y_n) converges as well and $\lim_{n\to\infty} y_n = L$.

4. Let the sequence $(x_n)_{n=1}^{\infty}$ be defined inductively by

$$x_1 = 2 \qquad \text{and} \qquad x_{n+1} = \frac{x_n}{2} + \frac{1}{4 - x_n}. \tag{$\star$}$$

   (a) Compute x_2 and x_3 (and maybe use a calculator to obtain approximations to the next few terms).

(b) Prove that $(x_n)_{n=1}^{\infty}$ is strictly decreasing. *Hint: the right hand side of* $(\star)$ *is monotonically increasing in the range* $(-\infty, 4)$.

(c) Prove that 0 is a lower bound for $(x_n)_{n=1}^{\infty}$.

(d) Deduce that $(x_n)_{n=1}^{\infty}$ converges and compute the limit. (This goes a little beyond where we are in the module, but do the best you can.)

5. For each of the following sequences, identify all its accumulation points. For each accumulation point $a \in \mathbb{R}$ give a subsequence that converges to a.

   (a) $(x_n)_{n=1}^{\infty}$, where $x_n = n^{-1}\cos(n\pi/2)$,

   (b) $(y_n)_{n=1}^{\infty}$, where $y_n = \cos(n\pi/2)$, and

   (c) $(z_n)_{n=1}^{\infty}$, where $z_n = n\cos(n\pi/2)$.

6. Consider the sequence $(x_n)_{n=1}^{\infty}$ defined by $x_n = (-1)^n(1 + 1/n)$. Define $b_k = \sup_{n \geq k} x_n = \sup\{x_n : n \in \mathbb{N} \text{ and } n \geq k\}$.

   (a) Evaluate the first six terms $b_1, b_2, \ldots, b_6$ of the sequence $(b_k)_{k=1}^{\infty}$, leaving the result as an exact rational number.

   (b) Give the general form for b_k, treating separately the cases k even and k odd.

   (c) Show that the sequence $(b_k)_{k=1}^{\infty}$ converges and state the limit of the sequence.

   (d) Verify that $(b_k)_{k=1}^{\infty}$ is decreasing.

7. Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers. Recall what it means for $(x_n)_{n=1}^{\infty}$ to be a *Cauchy sequence*.

   (a) Using only the definition, but not any results proved in the course, prove that $(x_n)_{n=1}^{\infty}$ given by
   $$x_n = 2 + \frac{1}{3n^2}$$
   is a Cauchy sequence.

   (b) Using only the definition, but not any results proved in the course, prove that $(x_n)_{n=1}^{\infty}$ given by
   $$x_n = \sum_{k=1}^{n} \frac{3}{k}$$
   is *not* a Cauchy sequence.

8. Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers and let $(y_n)_{n=1}^{\infty}$ be the sequence defined by $y_n = x_{n+1}$ for each $n \in \mathbb{N}$. Prove, using only the definition of convergence:

   (a) If $(x_n)_{n=1}^{\infty}$ converges to x, then $(y_n)_{n=1}^{\infty}$ converges to x.

   (b) If $(y_n)_{n=1}^{\infty}$ converges to x, then $(x_n)_{n=1}^{\infty}$ converges to x.