Random Processes – 2023/24 Assessment 1

This assessment consists of three questions. The focus is particularly on writing and explaining, including translating concepts between real-world descriptions and mathematics (in both directions).
Please submit your solutions to all questions via the module QMplus page. Work should be submitted as a handwritten and scanned (or electronically written on a tablet) document. The submission item on the QMplus page has more details and you should following the instructions there. You must submit your work by **5:00pm on Tuesday 7 November 2023**. Late work will not be accepted.
The work you submit must be your own. It is fine to discuss the problems with other students but you must write up your solutions yourself. Copying work from another person or getting someone else to do your work constitutes an assessment offence.

Further Instructions and Guidance

Read each question carefully and make sure that you answer the question you are asked and your answer is in the form required. There are three flavours of question on this sheet.

- Question involving mathematical explanation (Question 1(b), Question 2(a,c), Question 3).
- Questions involving numerical calculation (Question 1(a,c), Question 2(b)).
- Questions involving explanation of mathematical concepts in non-mathematical language (Question 1(d), Question 2(d,e)).

For the parts involving explanations, you should write in full sentences and your answer should be clear and accurate in both mathematics and use of English.
For the questions involving numerical calculations you will get full marks if your final answer is correct. However, you are advised to show working as this may give partial marks if the final answer is wrong.

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1. (40 marks) A machine can be in one of four states: ‘running smoothly’ (state 1), ‘running but needs adjustment’ (state 2), ‘temporarily broken’ (state 3), and ‘destroyed’ (state 4). Each morning the state of the machine is recorded. Suppose that the state of the machine tomorrow morning depends only on the state of the machine this morning subject to the following rules.

- If the machine is running smoothly, there is 1% chance that by the next morning it will have exploded (this will destroy the machine), there is also a 9% chance that some part of the machine will break leading to it being temporarily broken. If neither of these things happen then the next morning there is an equal probability of it running smoothly or running but needing adjustment.
- If the machine is temporarily broken in the morning then an engineer will attempt to repair the machine that day, there is an equal chance that they succeed and the machine is running smoothly by the next day or they fail and cause the machine to explode.
- If the machine is running but needing adjustment there is a 20% chance that an engineer will repair it so it is running smoothly the next day and otherwise it will remain in the same state for the next day.

Taking X_i to be the state of the machine on the morning of day i for $i \in \mathbb{N}$ we get a Markov chain which models the state of the machine.

(a) Write down the transition matrix for this Markov chain.

(b) The factory manager is interested in the number of days of smooth running we expect in the lifetime of the machine assuming that it starts its life running smoothly. Express this question in the Markov chain terminology we have developed in this module. Which Theorem in the notes can we use to calculate this?

(c) Calculate the expectation of the number of days of smooth running in the lifetime of the machine assuming that it starts its life running smoothly.

(d) The following mathematical expression describes an event. Give a description of this event as you would express it to a non-mathematician:

$$|\{n \in \{0, 1, \ldots, T-1\} : X_n = 1\}| > |\{n \in \{0, 1, \ldots, T-1\} : X_n = 2\}|$$

where $T = \min\{n : X_n = 4\}$.

2. (40 marks) Let $X_0, X_1, \ldots$ be the Markov chain on state space $\{1, 2, 3, 4\}$ with transition matrix

$$\begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/7 & 0 & 3/7 & 3/7 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 2/3 & 1/6 & 1/6 \end{pmatrix}$$

(a) Explain how you can tell this Markov chain has a limiting distribution and how you could compute it. Your answer should refer to the relevant Theorems in the notes.

(b) Find the limiting distribution for this Markov chain.

(c) Without doing any more calculations, what can you say about $p_{1,1}^{(100)}$ and $p_{2,1}^{(100)}$?

For the final two parts of this question, assume that this Markov chain is a model[1] of a country's economy. State 1 represents being 'in recession' and other states represent other broad categories of behaviour such as 'growing slowly'. The X_i (for $i \in \mathbb{N}$) represent the state the economy is in during the ith month after we started observing it.

(d) Give an interpretation of what the first entry of the distribution you found in part (b) tells you based on the definition of a limiting distribution. Your answer should be written for a non-mathematician and should consist of between 1 and 3 complete sentences without mathematical symbols or terminology.

(e) Give one other interpretation of the first entry of the distribution you found in part (b) written in the same style as for part (d).

3. (20 marks) Write down the Sheet number and Question number for one question (or part question) from Problem Sheets 1 to 4 that you were able to complete but only after a struggle. Say how you overcame the initial difficulty you had. Give a useful mathematical tip which you learnt from this struggle.

[Your answer to Question 3 should consist of between 3 and 5 complete sentences without mathematical symbols. There are lots of reasonable ways to interpret the last part of the question; in particular the tip could be quite general or quite specific.]

[1] probably a rather unrealistic one