Queen Mary
University of London

MTH5104: Convergence and Continuity 2023–2024
Problem Sheet 3 (Sequences 1)

1. Recall the Demon game arising from the definition of a sequence converging to 0: "Demon chooses $\epsilon > 0$, we choose $N \in \mathbb{N}$, Demon chooses $n > N$, ...". Suppose the Demon chooses $\varepsilon = 1/10$ in the first round. For each of the sequences below, state, with a brief justification, a winning N for us to choose in the second round:

   (a) $(x_n)_{n=1}^{\infty}$, where $x_n = 1/n$ for all n.
   (b) $(x_n)_{n=1}^{\infty}$, where $x_n = \cos(n\pi)/n$ for all n.
   (c) $(x_n)_{n=1}^{\infty}$, where $x_n = 1/\sqrt{n}$ for all n.
   (d) $(x_n)_{n=2}^{\infty}$, where $x_n = 1/\log_2 n$ for all $n \geq 2$.

2. As for Question 1, but now give a winning choice for the Demon in the first round, for the following sequences $(x_n)_{n=1}^{\infty}$ (this means that the sequence does *not* converge to 0). Again, briefly justify your answer.

   (a) $x_n = \frac{1}{4} - 1/n$,
   (b) $x_n = \frac{1}{4}\cos(n\pi)$, and
   (c) $x_n = \begin{cases} 1, & \text{if } n \text{ is a perfect cube;} \\ 0, & \text{otherwise.} \end{cases}$

3. For each of the following sequences state whether or not it converges to zero, and prove your answer. *Your proofs should be from "first principles": you should only use the definition of convergence as given in this course (Definition 3.1).* You must not assume any other results or techniques concerning sequences from this course, or from Calculus I or II. You *are* allowed to use the facts about real numbers proved in Chapter 2, such as the Archimedean property. If it helps, you should think first of the Demon game corresponding to convergence of the sequence, and who has the winning strategy.

   (a) $(x_n)_{n=1}^{\infty}$ given by $x_n = \frac{3}{n}$ for all n.
   (b) $(x_n)_{n=1}^{\infty}$ given by $x_n = 3$ for all n.
   (c) $(x_n)_{n=1}^{\infty}$ given by $x_n = \frac{5}{2n-1}$ for all n.
   (d) $(x_n)_{n=1}^{\infty}$ given by $x_n = \frac{3n+1}{n^2+1}$ for all n.
   (e) $(x_n)_{n=1}^{\infty}$ given by $x_n = \frac{1}{10}\sin(n\pi/2)$ for all n.

(f) $(x_n)_{n=1}^{\infty}$ given by $x_n = \begin{cases} 1, & \text{if } n \text{ is a power of 2;} \\ 0, & \text{otherwise.} \end{cases}$

(g) $(x_n)_{n=2}^{\infty}$ given by $x_n = 1/\log_2 n$ for all $n \geq 2$.

4. Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be two sequences which converge to zero and let $(z_n)_{n=1}^{\infty}$ be a sequence which does not converge to zero. We define the new sequences $(\tilde{x}_n)_{n=1}^{\infty}$, $(\tilde{y}_n)_{n=1}^{\infty}$, and $(\tilde{z}_n)_{n=1}^{\infty}$ as follows

$$\tilde{x}_n = \begin{cases} 100 & \text{if } n \leq 10, \\ x_n & \text{if } n > 10, \end{cases}$$

$$\tilde{y}_n = \begin{cases} n^3 & \text{if } n \leq 1000, \\ y_n & \text{if } n > 1000. \end{cases}$$

$$\tilde{z}_n = \begin{cases} 1/n & \text{if } n \leq 10^{10}, \\ z_n & \text{if } n > 10^{10}. \end{cases}$$

Prove that $(\tilde{x}_n)_{n=1}^{\infty}$ and $(\tilde{y}_n)_{n=1}^{\infty}$ converge to zero and that $(\tilde{z}_n)_{n=1}^{\infty}$ does not converge to zero.

5. For each of the following sequences state whether or not it converges to zero, and *prove your answer*. You may use any result from the lectures, *provided you state it clearly.*

(a) $(x_n)_{n=1}^{\infty}$ given by $x_n = \frac{5n+17}{n^2+3n}$.

(b) $(x_n)_{n=1}^{\infty}$ given by $x_n = \frac{1}{n+10\cos(\pi n)}$.

(c) $(x_n)_{n=1}^{\infty}$ given by $x_n = \frac{5n^2+17}{n^2+3n}$.

6. Suppose $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are sequences, and that $(x_n)_{n=1}^{\infty}$ converges to 0 and that $(y_n)_{n=1}^{\infty}$ does not. Let $z_n = x_n y_n$ for all $n \in \mathbb{N}$. By giving two examples, show that $(z_n)_{n=1}^{\infty}$ may or may not converge to 0. (You do not need to justify your examples as thoroughly as in the previous questions.)

7. Suppose $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are bounded sequences. Show that the sequence $(z_n)_{n=1}^{\infty}$ is bounded, where $z_n = x_n + y_n$ for all $n \in \mathbb{N}$.

8. For each of (a)–(d) give a sequence $(x_n)_{n=1}^{\infty}$ with the stated properties. In each case, the sequence $(y_n)_{n=1}^{\infty}$ is defined by $y_n = 2x_n^n$ for all $n \in \mathbb{N}$. Briefly explain your answers with respect to results and examples from the course.

(a) (x_n) converges to some $x \neq 0$, and (y_n) converges to some $y \neq 0$.

(b) (x_n) is bounded, but (y_n) does not converge.

(c) (x_n) does not converge (to any $x \in \mathbb{R}$), but (y_n) converges to zero.

(d) (x_n) does not converge (to any $x \in \mathbb{R}$), but (y_n) converges to $y \neq 0$.

9. **Challenge.** Suppose that the sequence $(x_n)_{n=1}^{\infty}$ converges to $x \neq 0$, and that $x_n \neq 0$ for all $n \in \mathbb{N}$. Prove that $(1/x_n)_{n=1}^{\infty}$ converges to $1/x$.

10. Show how to deduce part (iv) of Theorem 3.24 from Question 9.